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VARIATIONAL ANALYSIS OF AN ELASTIC PROBLEM INVOLVING TRESCA FRICTION LAW

AMINA AMASSAD, RALUCA IOANA ROSCA
and
MIRCEA SOFONEA

Abstract. This paper deals with the study of a quasistatic problem modelling the contact between an elastic body and a rigid foundation. The friction law considered here is the Tresca friction law. The paper presents two variational formulations of the problem followed by existence and uniqueness results. The link between the solutions of the previous variational problems is also studied, together with their mechanical interpretation.

1. Introduction

This work concerns the study of the quasistatic evolution of a linearly elastic body in bilateral contact with a rigid foundation. The friction law considered here is the Tresca friction law. The classical formulation of the mechanical problem is given by problem P. For this problem we propose two variational formulations given by P_1 and P_2. Problem P_1 is obtained from P using a Green type formula and the constitutive law. Since in this way the stress field is eliminated, the unknown of this problem is the displacement field denoted by u. Problem P_2 is also obtained from P, using a similar method. Since in this case the displacement field is eliminated, problem P_2 involves as unknown only the stress field denoted by σ. For problem P_1 we prove an existence and uniqueness result using a time discretization method (theorem 4.1). Under the same assumptions we also prove an existence and uniqueness result for the problem P_2, applying classical arguments of evolution equations (theorem 4.2). Finally, we study the link between the solution of the variational problems P_1 and P_2 (theorem 5.1). In particular, we obtain that if u denotes the solution of P_1 and σ represents the solution of P_2 then, denoting by $\varepsilon = \varepsilon(u)$ the small strain tensor, σ and ε must be related by the linear elastic constitutive law.

2. Problem statement and preliminaries

Let us consider an elastic body whose material particles fulfil a bounded domain $\Omega \subset \mathbb{R}^M$ $(M = 2, 3)$ and whose boundary Γ, assumed to be sufficiently smooth, is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3. Let $meas\ \Gamma_1 > 0$ and let $T > 0$ be a time interval. We suppose that the displacement field vanishes on $\Gamma_1 \times (0,T)$, that surface tractions F act on $\Gamma_2 \times (0,T)$ and that body forces b act in $\Omega \times (0,T)$. The solid is in contact with a rigid foundation on $\Gamma_3 \times (0,T)$ and it is submitted to friction forces. Assuming a linear elastic constitutive law and neglecting the inertial terms, the above mechanical problem may be formulated as follows :

Problem P : Find the displacement field $u : \Omega \times [0,T] \to \mathbb{R}^M$ and the stress field $\sigma : \Omega \times [0,T] \to \mathcal{S}_M$ such that :

$$\left\{\begin{array}{ll}
\sigma = \mathcal{E}\varepsilon(u) \quad in \ \ \Omega \times (0,T) & (2.1) \\[2mm]
Div\ \sigma + b = 0 \quad in \ \ \Omega \times (0,T) & (2.2) \\[2mm]
u = 0 \quad on \ \ \Gamma_1 \times (0,T) & (2.3) \\[2mm]
\sigma\nu = F \quad on \ \ \Gamma_2 \times (0,T) & (2.4) \\[6mm]
u_\nu = 0 \ , \ |\sigma_\tau| \leq g & \\[2mm]
|\sigma_\tau| < g \Rightarrow \dot{u}_\tau = 0 & \\[2mm]
|\sigma_\tau| = g \Rightarrow \ there\ exists\ \lambda \geq 0 \ \ such\ that \ \ \sigma_\tau = -\lambda \dot{u}_\tau & \\[2mm]
on \ \ \Gamma_3 \times (0,T) & (2.5) \\[6mm]
u(0) = u_0 \qquad in \ \ \Omega. & (2.6)
\end{array}\right.$$

Here $\mathcal{S}_M$ denotes the set of second order symmetric tensors on $\mathbb{R}^M$, $\mathcal{E}$ represents the fourth-order elastic tensor, $\varepsilon(u)$ denotes the small strain tensor and $Div\ \sigma$ represents the divergence of the tensor-valued function σ. In the previous problem ν denotes the unit outward normal to Ω, $\sigma\nu$ represents the Cauchy stress vector and u_0 is the initial data.

Let us also notice that (2.5) defines the Tresca friction law in which u_ν represents the normal displacement, $\dot{u}_\tau$ denotes the tangential velocity, σ_τ is the tangential force on the contact boundary and g is the friction yield limit. Moreover, in (2.5) as well as everywhere in this paper, the dot above represents the derivative with respect to the time variable.

We denote in the sequel by "." the inner product in the spaces $\mathbb{R}^M$ and $\mathcal{S}_M$ and by $|\cdot|$ the Euclidean norm of theses spaces. We also use the following notations :

$$H = \{\ v = (v_i) \mid v_i \in L^2(\Omega)\ ,\ i = \overline{1, M}\ \}\ ;$$

$$H_1 = \{ \ v = (v_i) \mid v_i \in H^1(\Omega) \ , \ i = \overline{1, M} \ \} \ ,$$
$$\mathcal{H} = \{ \ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega) \ , \ i, j = \overline{1, M} \ \} \ ,$$
$$\mathcal{H}_1 = \{ \ \tau \in \mathcal{H} \mid Div \ \tau \in H \ \} \ .$$

The spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the cannonical inner products denoted by $< .,. >_H$, $< .,. >_{H_1}$, $< .,. >_{\mathcal{H}}$ and $< .,. >_{\mathcal{H}_1}$.

Let now $v \in H_1$. For simplicity we shall use sometimes the same notation v for the trace of v on Γ and we denote by v_ν and v_τ respectively the normal and the tangential trace of v (see for instance [4],[8]). We also denote by V the closed subspace of H_1 given by

$$V = \{ \ v \in H_1 \mid \ v = 0 \quad \text{on} \ \ \Gamma_1 \ \ , \ \ v_\nu = 0 \quad \text{on} \ \ \Gamma_3 \ \}.$$

The deformation operator $\varepsilon : H_1 \to \mathcal{H}$ defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)) \quad , \quad \varepsilon(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

is a linear and continuous operator. Moreover, since $meas \ \Gamma_1 > 0$, Korn's inequality holds :

$$|\varepsilon(u)|_{\mathcal{H}} \geq C|u|_{H_1} \qquad \forall u \in V \tag{2.7}$$

(see for instance [7] p. 79). Here and everywhere in this paper C will represent strictely positive generic constants which may depend on Ω, Γ_1, Γ_2, Γ_3, $\mathcal{E}$, and T, and do not depend on time or on the input data.

On V we shall consider the inner product $< \cdot, \cdot >_V$ given by

$$< u, v >_V = < \varepsilon(u), \varepsilon(v) >_{\mathcal{H}} . \tag{2.8}$$

Using (2.7), we obtain that $| \cdot |_{H_1}$ and $| \cdot |_V$ are equivalent norms on V. Therefore, it results that $(V, | \cdot |_V)$ is a real Hilbert space.

Let us also recall that if $\tau \in \mathcal{H}_1$, we denote by τ_ν the normal trace of τ on Γ (see for instance [5] p.33) and that, if τ is a regular function, the following Green type formula holds :

$$\int_\Gamma \tau\nu \cdot v \ da = < \tau, \varepsilon(v) >_{\mathcal{H}} + < Div \ \tau, v >_H \qquad \forall v \in H_1. \tag{2.9}$$

3. Variational formulations

In this section we obtain two variational formulations for the mechanical problem P. For this, let us suppose that :

$$\begin{cases} \mathcal{E} : \Omega \times \mathcal{S}_M \to \mathcal{S}_M \quad \text{is a symmetric and positively definite} \\ \text{tensor, i.e. :} \\ (a) \ \mathcal{E}_{ijkh} \in L^\infty(\Omega) \quad \forall \ i, j, k, h = \overline{1, M} \\ (b) \ \mathcal{E}\sigma \cdot \tau = \sigma \cdot \mathcal{E}\tau \quad \forall \sigma, \tau \in \mathcal{S}_M, \ \text{a.e. in } \Omega \\ (c) \ \text{there exists} \ \ \alpha > 0 \ \ \text{such that} \ \ \mathcal{E}\sigma \cdot \sigma \geq \alpha|\sigma|^2 \quad \forall \sigma \in \mathcal{S}_M \end{cases} \tag{3.1}$$

$$b \in W^{1,\infty}(0,T,H) \quad , \quad F \in W^{1,\infty}(0,T,L^2(\Gamma_2)^M) \tag{3.2}$$

$$u_0 \in V \tag{3.3}$$

$$g \geq 0 \; . \tag{3.4}$$

Using (3.1) it results that $\mathcal{E}$ is an inversible tensor and we denote in the sequel by $\mathcal{E}^{-1}$ the inverse of $\mathcal{E}$. We also denote by $f : [0,T] \to V$ the function given by

$$< f(t), v >_V = \int_\Omega b(t).v dx + \int_{\Gamma_2} F(t).v \; da \quad \forall v \in V, \quad t \in [0,T]. \tag{3.5}$$

Using (3.2) and (3.5) it follows

$$f \in W^{1,\infty}(0,T,V). \tag{3.6}$$

Let us denote

$$\sigma_0 = \mathcal{E}\varepsilon(u_0) \tag{3.7}$$

and let j be the continuous seminorm on V defined by

$$j(v) = g \int_{\Gamma_3} |v_\tau| da \qquad \forall v \in V. \tag{3.8}$$

Finally, for all $t \in [0,T]$, let $\Sigma_{ad}(t)$ denote the set given by

$$\Sigma_{ad}(t) = \{\tau \in \mathcal{H} \mid \; < \tau, \varepsilon(v) >_{\mathcal{H}} + j(v) \;\geq\; < f(t), v >_V \quad \forall v \in V\} \tag{3.9}$$

and let us also suppose that

$$\sigma_0 \in \Sigma_{ad}(0). \tag{3.10}$$

We have the following result :

Theorem 3.1. If the couple of functions (u, σ) is a regular solution of the mechanical problem P then :

$$\begin{cases} u(t) \in V \quad , \quad < \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) >_{\mathcal{H}} + j(v) - j(\dot{u}(t)) \geq \\ \geq \; < f(t), v - \dot{u}(t) >_V \quad \forall v \in V \end{cases} \tag{3.11}$$

$$\sigma(t) \in \Sigma_{ad}(t) \quad , \quad < \tau - \sigma(t) \; , \; \varepsilon(\dot{u}(t)) >_{\mathcal{H}} \geq 0 \quad \forall \tau \in \Sigma_{ad}(t) \tag{3.12}$$

for all $t \in [0,T]$.

Proof. Let $v \in V$. Using (2.9) and (2.2) we have

$$< \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) >_{\mathcal{H}} = \; < b(t), v - \dot{u}(t) >_H + \int_\Gamma \sigma\nu \cdot (v - \dot{u}(t)) \; da \quad \forall t \in [0,T].$$

Using now (2.3),(2.4) and (3.5), from the previous equality we obtain

$$\begin{cases} < \sigma(t), \varepsilon(v) - \varepsilon(\dot{u}(t)) >_{\mathcal{H}} = \; < f(t), v - \dot{u}(t) >_V + \\ + \int_{\Gamma_3} \sigma\nu \cdot (v - \dot{u}(t)) da \quad \forall t \in [0,T]. \end{cases} \tag{3.13}$$

Moreover, from (2.5) and (3.8) it results

$$\int_{\Gamma_3} \sigma\nu \cdot (v - \dot{u}(t))da \geq j(\dot{u}(t)) - j(v) \quad \forall t \in [0, T]. \tag{3.14}$$

The inequality (3.11) follows now from (3.13) and (3.14).

Taking now $v = 2\dot{u}$ and $v = 0$ in (3.11) we obtain

$$< \sigma(t), \varepsilon(\dot{u}(t)) >_{\mathcal{H}} + j(\dot{u}(t)) \ = \ < f(t), \dot{u}(t) >_V \quad \forall t \in [0, T]. \tag{3.15}$$

Using again (3.11) and (3.15) it results

$$< \sigma(t), \varepsilon(v) >_{\mathcal{H}} + j(v) \ \geq \ < f(t), v >_V \quad \forall v \in V \ , \ t \in [0, T].$$

Hence, by (3.9) it follows

$$\sigma(t) \in \Sigma_{ad}(t) \quad \forall t \in [0, T]. \tag{3.16}$$

Let now $t \in [0, T]$ and let $\tau \in \Sigma_{ad}(t)$. We have

$$< \tau, \varepsilon(\dot{u}(t)) >_{\mathcal{H}} + j(\dot{u}(t)) \geq \ < f(t), \dot{u}(t) >_V . \tag{3.17}$$

The inequality (3.12) follows now from (3.15)-(3.17).

Having in mind the previous result as well as constitutive law (2.1) and the initial condition (2.6), we may consider the following variational problems :

Problem P_1 : Find the displacement field $u : [0, T] \to V$ such that :

$$\begin{cases} < \mathcal{E}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)) >_{\mathcal{H}} + j(v) - j(\dot{u}(t)) \geq \\ \geq \ < f(t), v - \dot{u}(t) >_V \quad \forall v \in V \ \ a.e. \ \ t \in (0, T) & (3.18) \\ u(0) = u_0 . & (3.19) \end{cases}$$

Problem P_2 : Find the stress field $\sigma : [0, T] \to \mathcal{H}$ such that :

$$\begin{cases} \sigma(t) \in \Sigma_{ad}(t) \quad \forall t \in [0, T] \ , \\ < \tau - \sigma(t), \mathcal{E}^{-1}(\dot{\sigma}(t)) >_{\mathcal{H}} \geq 0 \quad \forall \tau \in \Sigma_{ad}(t) \ \ , \ \ a.e. \ \ t \in (0, T) & (3.20) \\ \sigma(0) = \sigma_0 \ . & (3.21) \end{cases}$$

Let us now remark that problems P_1 and P_2 are formally equivalent to problem P. Indeed, if u represents a regular solution of the variational problem P_1 and σ is defined by $\sigma = \mathcal{E}\varepsilon(u)$, using the arguments of [3] it follows that (u, σ) is a solution of the mechanical problem P. Similarly, if σ represents a regular solution of the variational problem P_1 and $u \in V$ is defined by $\sigma = \mathcal{E}\varepsilon(u)$ then, using the same arguments, it follows that (u, σ) is a solution of the mechanical problem P. For this reason we may consider problems P_1 and P_2 as *variational formulations* of the mechanical problem P.

Finally, let us also recall that the study of the mechanical problem P using a different variational formulation involving a global state variable was already done in [6].

Under the assumptions (3.1)-(3.4), in the next section we present existence and the uniqueness results for the variational problems P_1 and P_2.

4. Existence and uniqueness results

The first result of this section is given by :

Theorem 4.1. Let (3.1)-(3.4) and (3.10) hold. Then there exists a unique solution of the problem P_1 having the regularity $u \in W^{1,\infty}(0,T,V)$.

Proof. The proof of theorem 4.1 may be obtained using similar arguments as in [1]. However, for the convenience of the reader, we present here a sketch of the proof. For this, let $N \in \mathbb{N}$, $h = \frac{T}{N}$, $t_n = nh$, $f_n = f(t_n)$ $\forall n = \overline{0,N}$. Let (u_n) be the sequence of elements of V given by

$$\begin{cases} < \mathcal{E}\varepsilon(u_{n+1}), \varepsilon(v) - \varepsilon(\frac{u_{n+1}-u_n}{h}) + j(v) - j(\frac{u_{n+1}-u_n}{h}) \geq \\ \geq < f_{n+1}, v - \frac{u_{n+1}-u_n}{h} >_V \quad \forall v \in V \ , \ n = \overline{0, N-1} \end{cases} \tag{4.1}$$

and let $u_N : [0,T] \to V$ be the function defined by

$$u_N(t) = \frac{t-t_n}{h}(u_{n+1} - u_n) + u_n \quad \forall t \in [t_n, t_{n+1}] \ , \ n = \overline{0, N-1}. \tag{4.2}$$

Using standard arguments of elliptic variational inequalities, from (4.1) we obtain that the sequence of functions (u_N) given by (4.2) is a bounded sequence in the space $W^{1,\infty}(0,T,V)$. So, by classical compactness arguments, it results that there exists $u \in W^{1,\infty}(0,T,V)$ such that, passing to a subsequence again denoted (u_N), we have :

$$u_N \rightarrow u \ \text{ in } \ L^\infty(0,T,V) \quad \text{weak} * \tag{4.3}$$

$$\dot{u}_N \rightarrow \dot{u} \ \text{ in } \ L^\infty(0,T,V) \quad \text{weak} * . \tag{4.4}$$

Having in mind (4.1)-(4.4) it results that u satisfies (3.18)-(3.19). Moreover, u is the unique solution of this problem.

The second result of this section is given by :

Theorem 4.2. Let (3.1)-(3.4) and (3.10) hold. Then there exists a unique solution of the problem P_2 having the regularity $\sigma \in W^{1,\infty}(0,T,\mathcal{H})$.

Proof. Let us firstly remark that (3.20) is equivalent to the nonlinear evolution equation

$$\mathcal{E}^{-1}\dot{\sigma}(t) \ + \ \partial\psi_{\Sigma_{ad}(t)}(\sigma(t)) \ni 0 \quad a.e. \ \ t \in (0,T) \tag{4.5}$$

where $\partial\psi_{\Sigma_{ad}(t)}$ denotes the subdifferential of the indicator function $\psi_{\Sigma_{ad}(t)}$ given by

$$\psi_{\Sigma_{ad}(t)}(z) = \begin{cases} 0 & \text{if } z \in \Sigma_{ad}(t) \\ +\infty & \text{if } z \notin \Sigma_{ad}(t). \end{cases}$$

Since the set $\Sigma_{ad}(t)$ depends on time, we shall replace (4.5) by a nonlinear evolution equation associated to a fixed convex set. For this, let us introduce the following notations :

$$\Sigma_0 = \{\tau \in \mathcal{H} \mid \ <\tau, \varepsilon(v)>_{\mathcal{H}} + j(v) \ \geq \ 0 \ \ \forall v \in V\} \tag{4.6}$$

$$\widetilde{\sigma} \ = \ \varepsilon(f) \tag{4.7}$$

$$\overline{\sigma} \ = \sigma - \widetilde{\sigma} \ , \ \ \overline{\sigma}_0 \ = \sigma_0 - \widetilde{\sigma}(0). \tag{4.8}$$

It is easy to see that σ is a solution of P_2 having the regularity $\sigma \in W^{1,\infty}(0,T,\mathcal{H})$ if and only if $\overline{\sigma} \in W^{1,\infty}(0,T,\mathcal{H})$ and

$$\begin{cases} \overline{\sigma}(t) \in \Sigma_0 \quad \forall t \in [0,T] \ , \\ <\tau - \overline{\sigma}, \mathcal{E}^{-1}\dot{\overline{\sigma}}(t)>_{\mathcal{H}} \geq 0 \quad \forall \tau \in \Sigma_0 \ , \quad a.e. \ on \ (0,T) & (4.9) \\ \overline{\sigma}(0) = \overline{\sigma}_0 \ . & (4.10) \end{cases}$$

Moreover, it results that the evolution problem (4.9)-(4.10) is equivalent to the following Cauchy problem

$$\begin{cases} \mathcal{E}^{-1}\dot{\overline{\sigma}}(t) \ + \ \partial\psi_{\Sigma_0}(\overline{\sigma}(t)) \ni 0 \quad a.e. \ \ t \in (0,T) & (4.11) \\ \overline{\sigma}(0) = \overline{\sigma}_0 & (4.12) \end{cases}$$

where $\partial\psi_{\Sigma_0}$ denotes the subdifferential of the indicator function ψ_{Σ_0}. Let us now remark that Σ_0 is a closed convex set in $\mathcal{H}$ and that from (3.10) and (4.8) we have $\overline{\sigma}_0 \in \Sigma_0$. Using now (3.1) and classical results of evolution equation (see for example [2] p. 189), we obtain the existence and the uniqueness of a function $\overline{\sigma} \in W^{1,\infty}(0,T,\mathcal{H})$ solution for (4.11)-(4.12), which proves theorem 4.2.

5. Equivalence results and mechanical interpretation

In this section we study the link between the solutions u and σ of the variational problems P_1 and P_2. The main result obtained here is the following :

Theorem 5.1. Let (3.1)-(3.4),(3.10) hold and let (u,σ) be a couple of functions such that $u : [0,T] \to V$ and $\sigma : [0,T] \to \mathcal{H}$. Let us also consider the properties :

i) u is the solution of the problem P_1 given in theorem 4.1 ;

ii) σ is the solution of the problem P_2 given in theorem 4.2 ;

iii) σ and u are connected by the elastic constitutive law

$$\sigma \ = \ \mathcal{E}\varepsilon(u) \quad \forall t \in [0,T]. \tag{5.1}$$

Then, if two of the previous properties are fulfilled, the remained one is also fulfilled.

Proof. $i) + iii) \Rightarrow ii)$ Let us suppose that $i)$ and $iii)$ hold. Using P_1 and (5.1) we obtain $\sigma \in W^{1,\infty}(0,T,\mathcal{H})$ and

$$< \sigma, \varepsilon(v) - \varepsilon(\dot{u}) >_{\mathcal{H}} + j(v) - j(\dot{u}) \geq < f, v - \dot{u} >_V \quad \forall v \in V \ , \ \ a.e. \ \ t \in (0,T). \quad (5.2)$$

Taking now $v = 2\dot{u}$ and $v = 0$ in (5.2) it follows

$$< \sigma, \varepsilon(\dot{u}) >_{\mathcal{H}} + j(\dot{u}) \ = < f, \dot{u} >_V \quad a.e. \ \ t \in (0,T). \quad (5.3)$$

So, from (5.2),(5.3) and having in mind the time regularity of σ and f, we obtain

$$\sigma(t) \in \Sigma_{ad}(t) \quad \forall t \in [0,T]. \quad (5.4)$$

Moreover, from (3.9) we obtain

$$< \tau, \varepsilon(\dot{u}) >_{\mathcal{H}} + j(\dot{u}) \ \geq < f, \dot{u} >_V \quad a.e \ . \ t \in (0,T). \quad (5.5)$$

Using now (5.3),(5.5),(5.1) and (3.7) we obtain that σ is the solution of the problem P_2.

$i) + ii) \Rightarrow iii)$ We suppose now that $i)$ and $ii)$ are fulfilled and let us introduce the function $\widetilde{\sigma}$ defined by

$$\widetilde{\sigma} = \mathcal{E}\varepsilon(u) \in W^{1,\infty}(0,T,\mathcal{H}). \quad (5.6)$$

As it results from the previous step, we obtain that $\widetilde{\sigma}$ is a solution of problem P_2. Using the uniqueness part of theorem 4.2 it results

$$\sigma = \widetilde{\sigma}. \quad (5.7)$$

So, from (5.7) and (5.6) we obtain $iii)$.

$ii) + iii) \Rightarrow i)$ Let now suppose that $ii)$ and $iii)$ hold. We have

$$< \tau - \overline{\sigma} \ , \ \mathcal{E}^{-1}\dot{\sigma} >_{\mathcal{H}} \geq 0 \quad \forall \tau \in \Sigma_0 \ \ , \ \ a.e. \ on \ (0,T) \quad (5.8)$$

where $\overline{\sigma} \in W^{1,\infty}(0,T,\mathcal{H})$ denotes the function given by (4.8),(4.7). Let us introduce the spaces W and $\mathcal{W}$ defined by

$$W \ = \ \{ \ v \in H_1 \ | \ \ v = 0 \ \ on \ \ \Gamma_1 \ \} \ ,$$
$$\mathcal{W} \ = \ \{ \ z \in \mathcal{H} \ | \ Div \ z = 0 \ \ in \ \ \Omega \ , \ z\nu = 0 \ \ on \ \ \Gamma_2 \cup \Gamma_3 \ \}.$$

For all $z \in \mathcal{W}$, using (2.9) and (4.6), we obtain $\overline{\sigma}(t) \pm z \in \Sigma_0 \ \ \forall \, t \in [0,T]$. Therefore, taking $\tau = \overline{\sigma} \pm z$ in (5.8), it results $< z, \mathcal{E}^{-1}\dot{\sigma} >_{\mathcal{H}} = \ 0 \ \ \forall z \in \mathcal{W}, \ a.e. \ on \ (0,T)$. Since the orthogonal complement in $\mathcal{H}$ of $\mathcal{W}$ is the space $\varepsilon(W)$ (see for instance [5] p. 34), having in mind the regularity $\sigma \in W^{1,\infty}(0,T,\mathcal{H})$ and the properties of the deformation operator $\varepsilon : W \to \varepsilon(W)$, there exists $\widetilde{v} \in L^{\infty}(0,T,W)$ such that

$$\mathcal{E}^{-1}\dot{\sigma} = \varepsilon(\widetilde{v}) \quad a.e. \ \ on \ \ (0,T). \quad (5.9)$$

We shall prove that $\widetilde{v} \in V \ \ a.e.$ on $(0,T)$. Indeed, using (5.8) and (5.9) we obtain

$$< \tau - \overline{\sigma} \ , \ \varepsilon(\widetilde{v}) >_{\mathcal{H}} \geq 0 \quad \forall \tau \in \Sigma_0 \ \ , \ \ a.e. \ \ on \ \ (0,T) \quad (5.10)$$

and let $t \in [0,T]$ such that

$$< \tau - \overline{\sigma}(t) \ , \ \varepsilon(\widetilde{v}(t)) >_{\mathcal{H}} \geq 0 \quad \forall \tau \in \Sigma_0. \quad (5.11)$$

Let us recall that, since $meas\ \Gamma_1 > 0$, W is a Hilbert space with the inner product $< u, v >_W = < \varepsilon(u), \varepsilon(v) >_{\mathcal{H}}$. Moreover, since V is a closed subspace of W, if $\widetilde{v}(t) \notin V$, there exists $\widetilde{\tau} \in \mathcal{H}$ such that

$$< \widetilde{\tau}, \varepsilon(v) >_{\mathcal{H}} = 0 \quad \forall v \in V \tag{5.12}$$

$$< \widetilde{\tau}, \varepsilon(\widetilde{v}(t)) >_{\mathcal{H}} < 0. \tag{5.13}$$

From (5.12) and (4.6) it follows that $\lambda\widetilde{\tau} \in \Sigma_0 \ \ \forall \lambda \geq 0$ and, using (5.11), we obtain

$$\lambda < \widetilde{\tau}, \varepsilon(\widetilde{v}(t)) >_{\mathcal{H}} \geq < \overline{\sigma}(t), \varepsilon(\widetilde{v}(t)) >_{\mathcal{H}} . \tag{5.14}$$

Using now (5.13) and passing to the limit in (5.14) when $\lambda \to +\infty$ we obtain a contradiction. So, it follows that the element $\widetilde{v}$ previously defined is such that $\widetilde{v} \in V$ *a.e.* on $(0, T)$ i.e. $\widetilde{v} \in L^\infty(0, T, V)$.

Taking now

$$\widetilde{u}(t) = \int_0^t \widetilde{v}(s)ds \ + \ u_0 \quad \forall t \in \ [0, T] \tag{5.15}$$

and using (3.3), we obtain $\widetilde{u} \in W^{1,\infty}(0, T, V)$. After integration, from (5.9) we obtain

$$\sigma(t) = \int_0^t \mathcal{E}\varepsilon(\widetilde{v}(s))ds \quad + \sigma_0 \quad \forall t \in [0, T],$$

and, using (3.7) and (5.15), it follows

$$\sigma(t) = \mathcal{E}\varepsilon(\widetilde{u}(t)) \quad \forall t \in [0, T]. \tag{5.16}$$

So, from (5.1) and (5.16) we have

$$\varepsilon(\widetilde{u}) \ = \ \varepsilon(u) \quad \forall t \in [0, T].$$

Having now in mind that $u(t) \in V$, $\widetilde{u}(t) \in V$ for all $t \in [0, T]$, using Korn's inequality (2.7) it results that $u = \widetilde{u}$ for all $t \in [0, T]$. Therefore $u \in W^{1,\infty}(0, T, V)$.

Using now the subdifferentiability of j on V and (2.8) it results that there exists $\overline{\tau} \ : \ [0, T] \to \mathcal{H}$ such that

$$< \overline{\tau}, \varepsilon(v) - \varepsilon(\dot{u}) >_{\mathcal{H}} \ + j(v) \ - \ j(\dot{u}) \geq < f, v - \dot{u} >_V \quad \forall v \in V, \ a.e. \ on \ (0, T). \tag{5.17}$$

Taking now $v = 2\dot{u}$ and $v = 0$ in (5.17), we have

$$< \overline{\tau}, \varepsilon(\dot{u}) >_{\mathcal{H}} \ + j(\dot{u}) \ = < f, \dot{u} >_V \quad a.e. \ on \ (0, T). \tag{5.18}$$

Using again (5.17) and (5.18) we obtain that $\overline{\tau} \in \Sigma_{ad}(t)$ *a.e. on* $(0, T)$ and, by *ii*) and *iii*), it results

$$< \overline{\tau}, \varepsilon(\dot{u}) >_{\mathcal{H}} \geq < \sigma, \varepsilon(\dot{u}) >_{\mathcal{H}} \qquad a.e. \ \ on \ \ (0, T). \tag{5.19}$$

Using now (5.19) and (5.18) we obtain

$$< f, \dot{u} >_V \geq < \sigma, \varepsilon(\dot{u}) >_{\mathcal{H}} \ + j(\dot{u}) \qquad a.e. \ \ on \ \ (0, T). \tag{5.20}$$

Finally, from (5.20), *ii*) and *iii*) we obtain that u is a solution of the problem P_1.

Mecanical interpretation. The mechanical interpretation of the results obtained in theorem 5.1 is the following :

1) if the displacement field u is the solution of the variational problem P_1 then the stress field σ associated to u by the elastic constitutive law $\sigma = \mathcal{E}\varepsilon(u)$ is the solution of the variational problem P_2.

2) if the stress field σ is the solution of the variational problem P_2 then the displacement field u associated to σ by the elastic constitutive law $\sigma = \mathcal{E}\varepsilon(u)$ is the solution of the variational problem P_1.

3) if the displacement field u is the solution of the problem P_1 and the stress field σ is the solution of the problem P_2 then u and σ are connected by the elastic constitutive law $\sigma = \mathcal{E}\varepsilon(u)$.

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Author's address :

Amina Amassad and Mircea Sofonea, Department of Mathematics, University of Perpignan, 52 Avenue de Villeneuve, 66860 Perpignan, France ;

Raluca Ioana Roşca, Department of Mathematics, University of Bucharest, Str. Academiei 14, 70109 Bucharest, Romania.