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SUFFICIENT CONDITIONS FOR THE COMPATIBILITY OF SOME SYSTEMS OF CONVEX INEQUALITIES

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Ky Fan studied in [2] the existence of solutions for some systems of convex inequalities involving lower semicontinous functions defined on a convex compact set in a topological vector space. [All the topological vector spaces considered in this paper, t.v.s. for short, are real and Hausdorff.] Particulary, he proved the following theorem:

THEOREM 1. *Let X be a nonempty convex compact subset of t.v.s. and let $\mathcal{F}$ be a family of convex lower semicontinuous functions $f: X \to]-\infty, \infty]$. Then the following conditions are equivalent:*

(i) *There exists an $x \in X$ such that:*

$$f(x) \le 0, \text{ for all } f \in \mathcal{F},$$

i.e., this system of inequalities is compatible.

(ii) *For any $n \in \mathbf{N}$, and $\alpha_i \ge 0$ with $\sum_{i=1}^{n} \alpha_i = 1$ and $f_1, f_2, \ldots, f_n \in \mathcal{F}$ there exists an x such that:*

$$\sum_{i=1}^{n} \alpha_i f_i(x) \le 0.$$

N. Shioji and W. Takahashi [6] and Shioji [5] extend the Ky Fan's theorem to so called "convexlike" functions with values in $]-\infty, \infty[$. Next we needed a special case of Theorem 1 in [6]:

COROLLARY 1. *Let X be a nonempty convex compact subset of t.v.s. and $\{f_1, f_2, \ldots, f_n\}$ be a finite family of convex lower semicontinuous functions $f_i : X \to]-\infty, \infty[$. Then the following conditions are equivalent:*

(i) *There exists an $x \in X$ such that:*

$$f_i(x) \le 0, \quad \forall i \in \{1, \ldots, n\}.$$

(ii) *For any $\alpha \in S_n$ there exists an $x \in X$ such that:*

$$\sum_{i=1}^{n} \alpha_i f_i(x) \le 0.$$

Here we denote by S_n the set:

$$S_n = \left\{ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{R}^n \colon \alpha_1 \geq 0, \ldots, \alpha_n \geq 0, \sum_{j=1}^{n} \alpha_j = 1 \right\}.$$

In this paper we obtain some sets of sufficient conditions for the compatibility of systems of convex inequalities (Theorems 2 and 3, and Corollary 2). As a by-product, we derive an intersectional result for minimal subfamilies of closed sets with convex complement (Theorem 4).

In our proofs we shall need the following lemmas proved in [1]:

LEMMA 1. *Let X be a nonempty convex compact subset of a t.v.s., k and l two positive integers with $k \leq l+1$, and $\mathcal{A}$ a family of closed convex subsets of X such that:*

(i) *For any subfamily $\mathcal{A}'$ of $\mathcal{A}$ which card $\mathcal{A}' = k$, we have $\cup \mathcal{A}' = X$.*

(ii) *For any subfamily $\mathcal{A}'$ of $\mathcal{A}$ which card $\mathcal{A}' = l$, we have $\cap \mathcal{A}' \neq \emptyset$.*

Then $\cap \mathcal{A} \neq \emptyset$.

LEMMA 2. *Let X be a nonempty convex subset of t.v.s., k and l two positive integers with $k \leq l+1$, and $\mathcal{A}$ a family of convex closed subsets of X satisfaying the conditions* (i) *and* (ii) *in* Lemma 1. *Then $\cap \mathcal{A} \neq \emptyset$.*

THEOREM 2. *Let X be a nonempty convex compact subset of a t.v.s., k and l two positive integers with $k \leq l+1$, and $\mathcal{F}$ a family of convex lower semicontinuous functions $f \colon X \to]-\infty, \infty]$ satisfaying the conditions:*

(a) *For any k functions $f_1, \ldots, f_k$ in $\mathcal{F}$ and any x in X there exists an α in S_k such that:*

$$\sum_{j=1}^{k} \alpha_j f_j(x) \leq 0.$$

(b) *For any l functions $f_1, \ldots, f_l$ in $\mathcal{F}$ and any α in S_l there exists an x in X such that:*

$$\sum_{j=1}^{l} \alpha_j f_j(x) \leq 0.$$

Then there exists an x in X such that:

$f(x) \leq 0$, for all $f \in \mathcal{F}$.

Proof. Denote by $\mathcal{A}$ the family of all sets $A_i = \{x \in X \colon f_i(x) \leq 0\}$ where $f_i \in \mathcal{F}$. Since the functions $f_i \in \mathcal{F}$ are convex and lower semicontinuous, the corresponding sets A_i are convex and closed in X. The proof of Theorem 2 will be achieved whenever we verify the conditions (i) and (ii) in Lemma 1 for the family $\mathcal{A}$.

If $\mathcal{A}$ does not satisfy the condition (i), than there are k functions $f_1, \ldots, f_k$ in $\mathcal{F}$ and an x in X such that $f_j(x) > 0$ for all $j \in \{1, 2, \ldots, k\}$. But in this case for any $\alpha \in S_k$ we have:

$$\sum_{j=1}^{k} \alpha_j f_j(x) > 0,$$ which contradicts condition (a).

Now, given a subfamily $\{A_1,...,A_l\}$ of l members in $\mathcal{A}$, i.e. $A_j = \{x \in X: f_j(x) \le 0\}, f_j \in \mathcal{F}$, then condition (b) together with Corollary 1 yield an x in X such that $f_j(x) \le 0$, $\forall j \in \{1,2,...,k\}$, i.e., $A_1 \cap ... \cap A_l \neq \emptyset$.

THEOREM 3. *Let X be a nonempty convex subset of t.v.s., k and l two positive integers, with $k \le l+1$, and $\mathcal{F}$ a finite family of convex lower semicontinuous functions $f: X \to]-\infty, \infty]$ satisfying the conditions:*

(i) *For any subfamily $\{f_1, ..., f_k\}$ of $\mathcal{F}$ and any $x \in X$ there is an $\alpha \in S_k$ such that:*

$$\sum_{j=1}^{k} \alpha_j f_j(x) \le 0.$$

(ii) *For any subfamily $\{f_1, ..., f_k\}$ of $\mathcal{F}$ there exists a compactsubset X_0 of X such that:*

$$\sum_{j=1}^{l} \alpha_j f_j(x) \le 0, \text{ for all } \alpha \in S_l \text{ and all } x \in X_0.$$

Then there exists an x in X such that $f(x) \le 0$ for all $f \in \mathcal{F}$.

Proof. Apply Lemma 2 to the family $\mathcal{A}$ of sets A_i in the proof of Theorem 2.

Another result concerning systems of convex inequalities we derive as application of the next intersection theorem.

THEOREM 4. *Let X be a nonempty convex subset of a t.v.s. and $\mathcal{A}$ a finite family of closed subsets of X having convex complements, i.e., $X \setminus A$ is convex for all $A \in \mathcal{A}$. If $\cup \mathcal{A} = X$ and $\cap \mathcal{A} = \emptyset$, that there exists a subfamily $\mathcal{A}'$ of $\mathcal{A}$ such that $\cup \mathcal{A}' = X$ and $\cap \mathcal{A}' \neq \emptyset$.*

Proof. Let $\mathcal{A}' = \{A_1,...,A_k\}$ be a minimal subfamily of $\mathcal{A}$ satisfying $\cup \mathcal{A}' = X$. To prove $\cap \mathcal{A}' \neq \emptyset$, suppose the contrary. Then the family $\{D_1,...,D_k\}$ is an open covering of X whenever we put $D_i = X \setminus A_i$. Denote by $\{p_1,...,p_k\}$ a continuous partition of unity corresponding to this covering, i.e., each $p_i: X \to [0,1]$ is a continuous function which vanishes outside of D_i and $\sum_{l=1}^{k} p_i(x) = 1$, for every $x \in X$ (see [3]).

Since $\{A_1,...,A_k\}$ is a minimal subfamily of $\mathcal{A}$ satisfying $A_1 \cup ... \cup A_k = X$, for each $j \in \{1,...,k\}$ there exists $x_j \in \cap\{D_i: i \in \{1,...,k\} \setminus \{j\}\}$. Now define the mapping $p: X \to X$ by:

$$p(x) = \sum_{i=1}^{k} p_i(x) \cdot x_i, \quad x \in X$$

and put $K = conv\{x_1,...,x_k\} \subset X$. Then p maps the nonempty convex compact set K into itself. Remark that K is homeomorphic with the closed unit ball of the Euclidian space $\mathbf{R}^n$, where $n \le k$ is the dimension of the vector subspace spanned by K [4, ch I, Th. 3.2], so that by Bronwer's fixed point theorem, there exists $z \in K$ such that $p(z) = z$.

Let $I = \{i \in \{1,...,k\}:\ p_i(z) > 0\}$ and $J = \{i \in \{1,...,k\}:\ p_i(z) = 0\}$. If $i \in I$ then $p_i(z) > 0$ hence $z \in \cap\{D_i;\ i \in I\}$. Furthermore, $J \subset \{1,...,k\} \setminus \{i\}$ whenever $i \in I$, hence by construction $x_i \in \cap\{D_j;\ j \in J\}$, so that the convexity of D_j implies:

$$p(x) = \sum_{i \in I} p_i(z) \cdot x_i \in \cap\{D_j;\ j \in J\}.$$

Therefore, $z = p(z) \in \cap\{D_i;\ i \in I \cup J\}$ which contradicts $\cup\{A_i;\ i \in I \cup J\} = X$.

COROLLARY 2. *Let X be a nonempty convex compact subset of t.v.s. and $\mathcal{F}$ a finite family of convex upper semicontinuous functions $f: X \to \mathbf{R}$ satisfying the conditions:*

(i) *For each $x \in X$ there is an $f \in \mathcal{F}$ such that $f(x) \geq 0$.*

(ii)*For each $x \in X$ there is an $g \in \mathcal{F}$ such that $g(x) < 0$.*

Then $\mathcal{F}$ contains a subfamily $\mathcal{F}'$ with the properties:

(i) *For each $x \in X$ there is an $f \in \mathcal{F}'$ such that $f(x) \geq 0$.*

(ii)There exists an *$x \in X$ such that $f(x) \geq 0$ for all $f \in \mathcal{F}'$.*

Proof. Apply Theorem 4 to the family $\mathcal{A}$ of all sets $A_f = \{x \in X:\ f(x) \geq 0\}$ associated with each $f \in \mathcal{F}$.

Remark. *If $X = \mathbf{R}^n$ in Corollary 2, by Helly's theorem it follows that card $\mathcal{F}' \leq n+1$.*

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