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APPROXIMATE METHOD FOR THE DYNAMIC RESPONSE DETERMINATION AT FORCED DAMPED VIBRATIONS

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ABSTRACT

This paper is intended to present an approximate method to determine the dynamic response of damping structures, beginning with the parameters value of the initial moment $t_0=0$ and assessing the value of these parameters at the moment t_1 t_0, based on some condition of Galerkin type. This method is easy to be simulated on the computer.

1. INTRODUCTION

In the structure dynamics the equation of motion under the most compact form is expressed by the differential equation in a matrix form [3]:

$$\mathbf{M}\cdot\ddot{\mathbf{u}}+\mathbf{B}\cdot\dot{\mathbf{u}}+\mathbf{R}\cdot\mathbf{u}=\mathbf{F}(t) \qquad (1)$$

with the initial conditions :

$$\begin{cases}\mathbf{u}(0)=\mathbf{u}_0\\ \dot{\mathbf{u}}(0)=\mathbf{v}_0\end{cases} \qquad (2)$$

where u(t) is vector of oscillation parameters, F(t) is vector of disturbing forces, M is masses matrix for n matrix points ($m_{ij}=0$ for $i\neq j$ and $m_{ii}=m_i$ for $i,j\in\{1,2,\ldots,n\}$), **B** is matrix of damping coefficients ($b_{ij}=b_{ji}$), **R** is matrix of structure rigidity ($r_{ij}=r_{ji}$).

It is required to find the solution of the differential system (1) with the initial conditions (2), that is to find the vector of displacement $\mathbf{u}(t)=(u_1(t), u_2(t), \ldots, u_k(t), \ldots, u_n(t))^T$, where $u_k(t)$ represents mass displacement m_k on the direction of oscillation parameters towards the equilibrium position.

2. EXPRESSION OF APPROXIMATE SOLUTION

It is considered the time range $[t_0,t_1]=[0,t_1]$ with $\Delta t=t_1-t_0$. Within this range the displacement vector **u**(t) is approximated by the polynomial vector:

$$\mathbf{u}(t)=\mathbf{A}_1+t\cdot\mathbf{A}_2+t^2\cdot\mathbf{A}_3+t^3\cdot\mathbf{A}_4,\quad t\in[0,\Delta t] \qquad (3)$$

where $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ and $\mathbf{A}_4$ are unknown vectors determined from the limited conditions of the range $[t_0,t_1]$ which are :

$$\begin{cases}\mathbf{u}(0)=\mathbf{u}_0 \ \text{and}\ \dot{\mathbf{u}}(0)=\mathbf{v}_0\\ \mathbf{u}(t_1)=\mathbf{u}_1 \ \text{and}\ \dot{\mathbf{u}}(t_1)=\mathbf{v}_1\end{cases} \qquad (4)$$

Vectors $\mathbf{u}_0$, $\mathbf{v}_0$ represent properly the initial conditions (2) and vectors $\mathbf{u}_\tau$ and $\mathbf{v}_\tau$ are considered parameters that should be determined from the conditions imposed on the rest.

Introducing the conditions (4) into the functions (3) it results :

$$\begin{cases} \mathbf{A}_1 = \mathbf{u}_0,\ \mathbf{A}_2 = \mathbf{v}_0, \\ \mathbf{A}_3 = \dfrac{2}{(\Delta t)^3}\cdot\mathbf{u}_0 - \dfrac{2}{(\Delta t)^3}\cdot\mathbf{u}_\tau + \dfrac{1}{(\Delta t)^2}\cdot\mathbf{v}_0 + \dfrac{1}{(\Delta t)^2}\cdot\mathbf{v}_\tau \\ \mathbf{A}_4 = -\dfrac{3}{(\Delta t)^2}\cdot\mathbf{u}_0 + \dfrac{3}{(\Delta t)^2}\cdot\mathbf{u}_\tau - \dfrac{2}{(\Delta t)}\cdot\mathbf{v}_0 - \dfrac{1}{(\Delta t)}\cdot\mathbf{v}_\tau \end{cases} \tag{5}$$

Replacing (5) in (3) and making the notations :

$$\begin{cases} \Phi_1(t) = 1 - \dfrac{3t^2}{(\Delta t)} + \dfrac{2t^3}{(\Delta t)^2};\ \Phi_2(t) = t - \dfrac{2t^2}{(\Delta t)} + \dfrac{t^3}{(\Delta t)^2} \\ \Phi_3(t) = \dfrac{3t^2}{(\Delta t)^2} - \dfrac{2t^3}{(\Delta t)^3};\quad \Phi_4(t) = -\dfrac{t^2}{(\Delta t)} + \dfrac{t^3}{(\Delta t)^2} \end{cases} \tag{6}$$

then the approximate solution is expressed under the form :

$$\mathbf{u}(t) = \Phi_1\cdot\mathbf{u}_0 + \Phi_2\cdot\mathbf{v}_0 + \Phi_3\cdot\mathbf{u}_\tau + \Phi_4\cdot\mathbf{v}_\tau \tag{7}$$

3. DETERMINATION OF THE REST

Introducing (7) in the differential system (1) it results the rest:

$$\boldsymbol{\varepsilon}(t) = \mathbf{M}\cdot\ddot{\mathbf{u}}(t) + \mathbf{B}\cdot\dot{\mathbf{u}}(t) + \mathbf{R}\cdot\mathbf{u}(t) - \mathbf{F}(t) \tag{8}$$

which can be expressed thus:

$$\boldsymbol{\varepsilon}(t) = \boldsymbol{\varphi}_1\cdot\mathbf{u}_0 + \boldsymbol{\varphi}_2\cdot\mathbf{v}_0 + \boldsymbol{\varphi}_2\cdot\mathbf{u}_\tau + \boldsymbol{\varphi}_3\cdot\mathbf{u}_\tau + \boldsymbol{\varphi}_4\cdot\mathbf{v}_\tau - \mathbf{F}(t) \tag{9}$$

where

$$\boldsymbol{\varphi}_k = \ddot{\Phi}_k\cdot\mathbf{M} + \dot{\Phi}_k\cdot\mathbf{B} + \Phi_k\cdot\mathbf{R};\quad k\in\{1,2,3,4\} \tag{10}$$

Introducing the notations:

$$\begin{cases} \mathbf{C}(t) = 2\cdot\mathbf{M} + 2t\cdot\mathbf{B} + t^2\cdot\mathbf{R} \\ \mathbf{D}(t) = t(6\mathbf{M} + 3t\cdot\mathbf{B} + t^2\cdot\mathbf{R}) \\ \mathbf{E}(t) = \mathbf{R}\cdot\mathbf{u}_0 + \mathbf{B}\cdot\mathbf{v}_0 + t\cdot\mathbf{R}\cdot\mathbf{v}_0 - \mathbf{F}(t) \end{cases} \tag{11}$$

and

$$\begin{cases} \mathbf{X} = \dfrac{1}{(\Delta t)^2}\left(-3\mathbf{u}_0 - 2(\Delta t)\cdot\mathbf{v}_0 + 3\mathbf{u}_\tau - (\Delta t)\cdot\mathbf{v}_\tau\right) \\ \mathbf{Y} = \dfrac{1}{(\Delta t)^3}\left(2\mathbf{u}_0 + (\Delta t)\cdot\mathbf{v}_0 - 2\mathbf{u}_\tau + (\Delta t)\cdot\mathbf{v}_\tau\right) \end{cases} \tag{12}$$

then the rest expression becomes:

$$\boldsymbol{\varepsilon}(t) = \mathbf{C}\cdot\mathbf{X} + \mathbf{D}\cdot\mathbf{Y} + \mathbf{E} \tag{13}$$

and its components are:

$$\varepsilon_i(t) = \sum_{j=1}^{n}\left(c_{ij}(t)\cdot x_j(t) + d_{ij}(t)\cdot y_j(t) + e_i(t)\right) \text{ for } i\in\{1,2,\ldots,n\}. \tag{13'}$$

4. DETERMINATION OF VECTORS $\mathbf{u}_\tau$ and $\mathbf{v}_\tau$

Within the time range $[0,\Delta t]$, $\Phi_3(t)$ and $\Phi_4(t)$ are considered as basic functions as they keep the sign constant within the range. To find the vectors $\mathbf{u}_\tau$ and $\mathbf{v}_\tau$ the conditions of the rest ortogonalization (13') are assessed with basic function $\Phi_3(t)$ and $\Phi_4(t)$ within the time range $[0,\Delta t]$

$$\begin{cases}\langle \Phi_3, \varepsilon_i(t)\rangle = 0 \\ \langle \Phi_4, \varepsilon_i(t)\rangle = 0\end{cases} \quad \text{for } i\in\{1,2,\ \dots\ ,n\}. \tag{14}$$

Introducing the notations:

$$\begin{cases} a_{i,j} = \int_0^{\Delta t} c_{ij}(t)\cdot\Phi_3(t)dt; \quad a_{i,j+n} = \int_0^{\Delta t} d_{ij}(t)\cdot\Phi_3(t)dt\ . \\ a_{n+i,j} = \int_0^{\Delta t} c_{ij}(t)\cdot\Phi_4(t)dt; \ a_{n+i,j+n} = \int_0^{\Delta t} d_{ij}(t)\cdot\Phi_4(t)dt \\ g_i = -\int_0^{\Delta t} e_{ij}(t)\cdot\Phi_3(t)dt; \quad g_{n+i} = -\int_0^{\Delta t} e_i(t)\cdot\Phi_4(t)dt \end{cases} \tag{15}$$

where $\begin{cases} A = (a_{k,l}) \quad k,l\in\{1,2,\dots,2n\} \\ G = (g_1, g_2,\dots g_{2n})^T \\ Z = (x_1, x_2,\dots,x_n, y_1, y_2,\dots,y_n)^T \end{cases}$

then the system (14), taking into account (13'),may be written under the compact form:

$$\mathbf{A}\cdot\mathbf{Z} = \mathbf{G} \tag{16}$$

Solving the system (16) it results vector **Z**, therefor the vectors **X** and **Y** are determined. Taking into account the relations (12) it results the system:

$$\begin{cases} 3\mathbf{u}_\tau-(\Delta t)\cdot\mathbf{v}_\tau = (\Delta t)^2\cdot\mathbf{X} + 3\ \mathbf{u}_0 + 2\cdot(\Delta t)\cdot\mathbf{v}_0 \\ -2\ \mathbf{u}_\tau + (\Delta t)\cdot\mathbf{v}_\tau = (\Delta t)^3\cdot\mathbf{Y} - 2\mathbf{u}_0 - (\Delta t)\cdot\mathbf{v}_0 \end{cases} \tag{17}$$

Noting:

$$\begin{cases} \mathbf{H}_1 = (\Delta t)^2\cdot\mathbf{X} + 3\mathbf{u}_0 + 2(\Delta t)\cdot\mathbf{v}_0 \\ \mathbf{H}_2 = (\Delta t)^3\cdot\mathbf{Y} - 2\mathbf{u}_0 - (\Delta t)\cdot\mathbf{v}_0 \end{cases} \tag{18}$$

the system (17) is written as follows:

$$\begin{cases} 3\mathbf{u}_\tau - (\Delta t)\cdot\mathbf{v}_\tau = \mathbf{H}_1 \\ -2\mathbf{u}_\tau + (\Delta t)\cdot\mathbf{v}_\tau = \mathbf{H}_2 \end{cases} \tag{17'}$$

Solving (17') it results the parameters vectors $\mathbf{u}_\tau$ and $\mathbf{v}_\tau$

$$\begin{cases} \mathbf{u}_\tau = \mathbf{H}_1 + \mathbf{H}_2 \\ \mathbf{v}_\tau = \dfrac{1}{\Delta t}\left(2\mathbf{H}_1 + 3\mathbf{H}_2\right) \end{cases} \tag{19}$$

By mean of these vectors one may write the system solution (1) in point $t = t_1$ by replacing in (7):

$$\mathbf{u}(t) = \Phi_1\cdot\mathbf{u}_0 + \Phi_2\cdot\mathbf{v}_0 + \Phi_3\cdot(\mathbf{H}_1 + \mathbf{H}_2) + \Phi_4\cdot(2\mathbf{H}_1 + 3\mathbf{H}_2)/\Delta t \tag{20}$$

This is the approximate solution within the time range $[t_0, t_1] = [0, \Delta t]$.

5. CALCULS OF MATRICES A AND G

To determine the coefficients of the matrices **A** and **G** one may use an approximate method for integral calculus or may use the following idea:

Functions $\Phi_3(t)$ and $\Phi_4(t)$ may be written under the form:

$$\Phi(\alpha_1, \alpha_2, t) = \alpha_1\cdot t^2 + \alpha_2\cdot t^3 \tag{21}$$

thus:

$$\begin{cases} \Phi_3(t) = \Phi\left(\dfrac{3}{(\Delta t)^2}, \dfrac{-2}{(\Delta t)^3}, t\right) \\ \Phi_4(t) = \Phi\left(-\dfrac{1}{\Delta t}, \dfrac{1}{(\Delta t)^2}, t\right) \end{cases} \tag{22}$$

Analogously, functions $c_{ij}(t)$ and $d_{ij}(t)$ given by (11) may be written by mean of the function:

$\Psi(\beta_1, \beta_2, \beta_3, \beta_4, t) = \beta_1 + \beta_2 \cdot t + \beta_3 \cdot t^2 + \beta_4 \cdot t^3$ (23)

thus: $\begin{cases} c_{ij} = \Psi(2m_{ij}, 2b_{ij}, r_{ij}, 0, t) \\ d_{ij} = \Psi(0, 6m_{ij}, 3b_{ij}, r_{ij}, t) \end{cases}$ for $\forall$ i, j$\in$\{1,2,…,n\} (24)

To obtain more compact and accurate expressions for the calculation of the coefficients a_{ij} and g_i with i, j$\in$\{1,2,…,n\} one may write as follows:

$$I(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) = \int_0^{\Delta t} \Phi(\alpha_1, \alpha_2, t) \cdot \Psi(\beta_1, \beta_2, \beta_3, \beta_4) dt \qquad (25)$$

Taking into account (21) and (23) then:

$$I(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \beta_4) = \frac{\alpha_1 \cdot \beta_1}{3} \cdot (\Delta t)^3 + \frac{\alpha_1 \beta_2 + \alpha_2 \beta_1}{4} \cdot (\Delta t)^4 + \\ + \frac{\alpha_1 \beta_3 + \alpha_2 \beta_2}{5} \cdot (\Delta t)^5 + \frac{\alpha_1 \beta_4 + \alpha_2 \beta_3}{6} \cdot (\Delta t)^6 + \frac{\alpha_2 \beta_4}{7} \cdot (\Delta t)^7 \qquad (25')$$

By mean of (25) and of the expression (25') the integrals (15) may be calculated accurately as follows:

$$\begin{cases} a_{i,j} = I\left(\frac{3}{(\Delta t)^2}, \frac{-2}{(\Delta t)^3}, 2m_{ij}, 2b_{ij}, r_{ij}, 0\right) \\ a_{i,j+n} = I\left(\frac{3}{(\Delta t)^2}, \frac{-2}{(\Delta t)^3}, 0, 6m_{ij}, 3b_{ij}, r_{ij}\right) \\ a_{i+n,j} = I\left(-\frac{1}{\Delta t}, \frac{1}{(\Delta t)^2}, 2m_{ij}, 2b_{ij}, r_{ij}, 0\right) \\ a_{n+i,n+j} = I\left(-\frac{1}{\Delta t}, \frac{1}{(\Delta t)^2}, 0, 6m_{ij}, 3b_{ij}, r_{ij}\right) \end{cases} \quad \text{for } i, j \in \{1,2,\dots,n\} \qquad (26)$$

Turning back to the vector **E**(t) (11), its components may defined by

$e_i(t) = h_i + t \cdot k_i - F_i(t)$ $\quad(\forall)$ i $\in$\{1,2,…,n\} (27)

where

$$\begin{cases} h_i = \sum_{j=1}^{n} \left(r_{ij} \cdot u_j^0 + b_{ij} \cdot v_j^0\right) \\ k_i = \sum r_{ij} \cdot v_j^0 \end{cases} \quad \text{for } i \in \{1,2,\dots,n\} \qquad (28)$$

The coefficients of the free from the system (16) are calculated with the relation (15) and may be explained as follows:

$$\begin{cases} g_i = -\int_0^{\Delta t} e_i(t) \cdot \Phi_3(t) dt = -L_i + J_i \\ g_{i+n} = -\int_0^{\Delta t} e_i(t) \cdot \Phi_4(t) dt = -L_{i+n} + J_{i+n} \end{cases} \qquad (29)$$

where:

$$\begin{cases} L_i = \int_0^{\Delta t} (h_i + k_i \cdot t) \cdot \Phi_3(t) dt = I\left(\frac{3}{(\Delta t)^2}, \frac{-2}{(\Delta t)^3}, h_i, k_i, 0, 0\right) \\ L_{i+n} = \int_0^{\Delta t} (h_i + k_i \cdot t) \cdot \Phi_4(t) dt = I\left(-\frac{1}{\Delta t}, \frac{1}{(\Delta t)^2}, h_i, k_i, 0, 0\right) \end{cases} \qquad (30)$$

and

$$\begin{cases} J_i = \int_0^{\Delta t} F_i(t+t_0)\cdot\Phi_3(t)dt \\ J_{i+n} = \int_0^{\Delta t} F_i(\ t+t_0)\cdot\Phi_4(t)dt \end{cases} \quad \text{for } i \in \{1,2,\ldots,n\} \tag{31}$$

By mean of the integral (25') one may accurately calculate the coefficients of the system (6), excepting the free terms which may be calculated with (30) and (31).

To calculate the integrals given by (31) an approximate method may be used, which may have as small as error for each type of disturbing forces.

6. ERROR OF METHOD

If we consider the method as an approximate one, which is the error made when calculating the solution (1) with the initial conditions (2) within the time range [0,t].

When noting with $\overline{\mathbf{u}}(t)$ the accurate solution of the problem (1) with (2), and noting with $\mathbf{u}(t)$ the approximate solution given by (20), it follows that the error in a point $t \in (0,t)$ is:

$$\boldsymbol{\delta}(t) = \mathbf{u}(t) - \overline{\mathbf{u}}(t) \qquad (\forall)\ t \in [0,t] \tag{32}$$

and the total absolute error within the range $[0,t] = [0,\Delta t]$ is:

$$\boldsymbol{\delta}^{*}(t) = \int_0^{\Delta t} \left|\delta(t)\right| dt \tag{33}$$

In [5] it is settled that this error increased on each component thus:

$$\delta_i^{*}(t) \le \mathbf{H}_i\,(\Delta t)^2 \tag{33'}$$

where

$$\mathbf{H}_i = \max\left\{\frac{7}{20}\left|\dot{u}_i(\theta)\right| + \frac{5}{12}\left|\dot{u}_i^{\,0}\right| + \frac{1}{4}\left|\dot{u}_i^{\,t}\right| + \frac{\Delta t}{2}\left|\ddot{u}_i(\xi)\right|\right\} \quad \text{for } \theta, \xi \in (0, \Delta t) \tag{34}$$

for $\forall\ i \in \{1, 2, \ldots, n\}$.

7. ALGORITHM FOR SOLUTION DETERMINATION

To determine the approximate solution (20) for any time range $[0, \Delta t]$ (with a small Δt) it is necessary to find the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$. For this determination one should pass through the following steps:

- -calculate matrix **A** using (28);
- -calculate vector **L** using (30);
- -calculate vector **J** using (31);
- -calculate vector **G** given by (29);
- -solve the system (16) and get the vectors **X** and **Y**;
- -determine the vectors $\mathbf{H}_1$ and $\mathbf{H}_2$ given by (18);
- -find the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ by mean of (19);
- -write the approximate solution within the range $[0, \Delta t]$.

The smaller the time range is the smaller the error is, according to (33) and thus the approximate solution is closer to the accurate solution.

8. CONCLUSIONS

a) Problem (1) with (2) has appeared during the study of mechanical and acoustic vibrations and in electronics, and it is solved for the case $\mathbf{B} = \mathbf{0}$. When $\mathbf{B} \neq \mathbf{0}$, the existent modal method needs the introducing of some approximate dumping factors on each component [3]

b) This method gives the solution of the problem (1) with (2), with the special accuracy if $\Delta t \in \{0.1, \ldots, 0.001\}$.

c) This method is applicable for any type of disturbing force $\mathbf{F}(t)$ and for any type of dumping **B**

d) The method may be use for any time range [0,T] as follows:

-Divide time range into n subranges, equidistant or not $[t_i, t_{i+1}]$ with $\Delta t = t_{i+1} - t_i$ and $i \in \{0, 1, \ldots, m-1\}$.

-Apply the algorithm for the time range $[t_1, t_2]$ and it leads to $\mathbf{u}(t_2)$ and $\mathbf{v}(t_2)$.

-Repeat the operation, analogously, for the time range $[t_{m-1}, t_m]$ resulting $\mathbf{u}(t_m)$ and $\mathbf{v}(t_m)$.

c) the algorithm presented above should by easily simulated on the computer due to the calculus systematization.

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