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CONDITION FOR VIBRATIONS STABILITY DAMPED DYNAMIC SYSTEMS

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1. ABSTRACT

In the present paper the necessary conditions are assessed for the polynomial attached to the equation of damped eigenpulsations to have the complex roots with the real part negative. This assures the stability of the system at free vibrations. In this way there are introduced notions of quasi-stability and stability, for the dynamic systems, by mean of Hurwitz polynomials.

2. INTRODUCTION

The dynamic computation of the structures is accomplished, in almost all cases, on the basis of the same dynamic models, with n discreet masses, constituting systems with finite number of degrees of freedom. From a dynamic system with n degrees of freedom, the equation of motion [2] is written under the form of matrix:

$$\mathbf{M}\cdot\ddot{\mathbf{u}}+\mathbf{B}\cdot\dot{\mathbf{u}}+\mathbf{R}\cdot\mathbf{u}=\mathbf{F}(t) \qquad (1)$$

where **M** is mass matrix ($m_{ij}=0$ for $i \neq j$, $m_{ii}=m_i$), **B** is damping coefficients matrix ($b_{ij}=b_{ji}$), **R** is rigidity coefficients matrix ($r_{ij}=r_{ji}$); **u** is vector of oscillation parameters, **F** is vector of disturbing forces depending on the direction of oscillation parameters.

The characteristic equation of undamped eigenpulsations ($\mathbf{B}=0$) is:

$$\det(\mathbf{R}-p^2\cdot\mathbf{M})=0 \qquad (2)$$

and it has real and positive roots:

$$p_1<p_2<\ldots<p_k<\ldots<p_n. \qquad (3)$$

The smallest eigen value p_1, is called fundamental eigenpulsation . Eigenvectors $\mathbf{V}_1,\mathbf{V}_2,\ldots,\mathbf{V}_n$ corresponding to eigenvalues $p_1, p_2, \ldots, p_n$ are grouped into the modal matrix $\mathbf{V}=[\mathbf{V}_1\ \mathbf{V}_2 \ldots \mathbf{V}_n]$.

In the case of the damped structure($\mathbf{B}\neq0$)one may admit that $\mathbf{B}=\gamma_m\cdot\mathbf{M}+\gamma_r\cdot\mathbf{R}$ and then the dynamic response through the modal method is expressed according to the modal matrix and the damping factors corresponding to each eigenform (experimental determined)[2].

When the damping matrix **B** is neither diagonal not proportional with **M** and **R** ,then in [4] it is described a numerical method to determine the damped eigenpulsations, as being the real part of the characteristic equation roots:

$$\Delta(\lambda)=\det(\lambda^2\cdot\mathbf{M}+\lambda\cdot\mathbf{B}+\mathbf{R})=0 \qquad (4)$$

For the structure to perform the free periodic damping vibrations, the equations(4)of the degree 2n in λ must have complex conjugate roots with the real part negative:

$$\lambda_k = a_k + ib_k;\ a_k < 0 \text{ for } k \in \{1,2,\ldots,n\} \quad (5)$$

Damped eigenpulsations are:

$$p_k^a = b_k, \qquad k \in \{1,2,\ldots,n\} \quad (6)$$

Between the roots of equation (2) and those of the equation (4) it is the relation:

$$|\lambda_k| = p_k, \qquad \text{for } k \in \{1,2,\ldots,n\} \quad (7)$$

For the equation (4) to have roots of the form (5), the matrices **M**,**B** and **R** must accomplish certain conditions, which will bee assessed further. In the structure already built, what one may modify is only the damping matrix **B.**

3. STABILITY CONDITIONS FOR DYNAMIC SYSTEMS

In the paper [4] it is shown that the equation (4) may be written under the form of a polynomial of the degree $n\cdot 2$.

$$\Delta(\lambda) = a_0\cdot \lambda^{2n} + a_1\cdot \lambda^{2n-1} + a_2\cdot \lambda^{m-2} + \ldots + a_{2n-1}\cdot \lambda + a_{2n} = 0 \quad (8)$$

Evidently, the roots of equation (8) are the same as those of equation (4).

DEFINITION 1. Dynamic structure characterized by the matrices **M**,**B** and **R** is quasi-stable, if the roots of equation (8) are complex conjugate roots and have the real part negative.
This means that they are all situated on the left side of the axis Oy.

In the theory of the automatic systems, the notions of quasi-stability is equivalent with the notion of asymptotic stability [3].

If the equation $\Delta(\lambda) = 0$ has all the roots with the real part strictly negative, then the polynomial $\Delta(\lambda)$ is called Hurwitz polynomial [3],[5]. To the characteristic polynomial (8) one may attach the matrix of Hurwitz:

$$H_{2n} = \begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_{2n} & a_{2n-1} & a_{2n-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_{2n} \end{bmatrix} \quad (9)$$

and the submatrices H_k, for $k \in \{1,2,\ldots,2n-1\}$ where H_k, contains the first k columns and the k lines from the matrix H_{2n} for $k \in \{1,2,\ldots,2n-1\}$. The notion of quasi-stability is equivalent only with the condition that the characteristic polynomial $\Delta(\lambda)$ to be hurwitzian.

THEOREM 1. The characteristic polynomial $\Delta(\lambda)$ is hurwitzian if and only if

$$a_0 > 0,\ a_1 > 0, \ldots,\ a_{2n} > 0 \qquad \det H_1 > 0,\ \det H_2 > 0,\ \ldots.\ \det H_{2n} > 0. \quad (10)$$

The conditions (10) are necessary and sufficient for the characteristic polynomial $\Delta(\lambda)$ to have the roots situated within the semiplane $\mathrm{Re}(\lambda) < 0$, but it is not specified that these roots may not be real or negative. That, for the structure, should mean a non-periodic movement (non-vibrator). We are interested to find some conditions to be accomplished by the huewitzian polynomial $\Delta(\lambda)$, so that it should not have the real roots negative.

The equation $\Delta(\lambda) = 0$ has the same roots as the equation $2\cdot\Delta(\lambda) = 0$, and the polynomial $2\cdot\Delta(\lambda)$ may be written as follows:

$$2\cdot\Delta(\lambda) = (2a_0\cdot\lambda^2 + 2a_1\cdot\lambda + a_2)\cdot\lambda^{2n-2} + (a_2\cdot\lambda^2 + 2a_3\cdot\lambda + a_4)\cdot\lambda^{2n-4} + \ldots + (a_{2n-2}\cdot\lambda^2 + 2a_{2n-1}\cdot\lambda + 2a_{2n}).(11)$$

For the polynomial (11) the following conditions are introduced:

$$\begin{aligned} &2a_0\cdot\lambda^2 + 2a_1\cdot\lambda + a_2 > 0 \\ &a_2\cdot\lambda^2 + 2a_3\cdot\lambda + a_4 > 0 \\ &a_4\cdot\lambda^2 + 2a_5\cdot\lambda + a_6 > 0 \\ &\vdots \\ &a_{2n-4}\cdot\lambda^2 + 2a_{2n-3}\cdot\lambda + a_{2n-2} > 0 \\ &a_{2n-2}\cdot\lambda^2 + 2a_{2n-1}\cdot\lambda + a_{2n} > 0 \end{aligned} \quad (12)$$

for ($\forall$) $\lambda \in \mathbf{R}$, then the equation $2 \cdot \Delta(\lambda) = 0$ may have neither real roots and, therefore, nor real negative roots.

The conditions (12) are equivalent to:

$$a_0 > 0,\ a_2 > 0, \ldots,\ a_{2n-2} > 0,\ a_{2n} > 0 \text{ and}$$

$$\begin{aligned} & a_1^2 < 2 \cdot a_0 \cdot a_2 \\ & a_3^2 < a_2 \cdot a_3 \\ & a_5^2 < a_4 \cdot a_6 \\ & \vdots \\ & a_{2n-3}^2 < a_{2n-4} \cdot a_{2n-2} \\ & a_{2n-1}^2 < 2 \cdot a_{2n-2} \cdot a_{2n} \end{aligned} \qquad (13)$$

If the conditions (13) are accomplished, then the equation $2 \cdot \Delta(\lambda) = 0$ may not have real negative roots. Using the results (10) and (13) one may introduce the notion of stable dynamic system for the dynamic structures.

DEFINITION 2. The dynamic system characterized by the matrices **M**, **B** and **R** is stable if it is quasi-stable and the characteristic equation (8) has no real negative roots.

THEOREM 2. If the dynamic system characterized by the matrices **M, B** and **R** which have characteristic equation (8) satisfies the conditions (10) and (13), then that system is stable. When a structure was designed and started up (matrices **M** and **R** locked) and when in operation vibrations with large amplitude have appeared, then the dynamic response may be modelled by introducing damping (matrix **B**). The values of these damping (matrix **B**) must be chosen, so that the dynamic system to be stable; that means the system should perform a vibratory movement. Therefore, the theory developed permits a simulation of the damping, that makes possible an optimization of the dynamic response of the structure.

4. APPLICATIONS

It is considered the dynamic system for which it is known: m = 1000[kg], K=1000[kN/m] and b = 50 · k_1 [N.S/m], where k_1 is a parameter.

The matrices of the equation of motion are:

$$\mathbf{M} = \begin{pmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 1.5m \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b & -b & 0 \\ -b & 3b & -2b \\ 0 & -2b & 3b \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} 6k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & 4k \end{pmatrix}$$

The characteristic equation of the undamped eigenpulsations is given by (2)and it is:

$$p^6 - 8{,}6(6)p^6 + 22{,}(3)p^2 - 16{,}(6) = 0 \qquad (14)$$

which has undamped eigenpulsations:

$$p_1 = 1{,}147754 s^{-1};\ p_2 = 1.6589 s^{-1};\ p_3 = 2.14414\ s^{-1}$$

Case a) For $k_1 = 1$ the characteristic equation (8) of the eigenpulsations is:

$$\Delta(\lambda) = \lambda^6 + 0{,}275\lambda^5 + 8{,}68\lambda^4 + 1{,}3584\lambda^3 + 22{,}3591\lambda^2 + 1{,}41669\lambda + 16{,}(6) = 0 \qquad (15)$$

where $a_0 = 0$; $a_1 = 0{,}275$; $a_2 = 8{,}868$; $a_3 = 1{,}3584$; $a_4 = 22{,}3591$; a5 = 1,41669; a6 = 16.(6).

Hurwitz matrix extracted from the polynomial is:

$$H_6 = \begin{pmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \\ 0 & a_6 & a_5 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_6 & a_5 & a_4 \\ 0 & 0 & 0 & 0 & 0 & a_6 \end{pmatrix} \qquad (16)$$

and $\det H_1 = a > 0$; $\det H_2 = 1{,}0286 > 0$; $\det H_3 = 0{,}096 > 0$

$$\det H_4 = 0{,}9138 > 0;\ \det H_5 = 0{,}0256 > 0;\ \det H_6 = 0{,}42667 > 0 \qquad (17)$$

The conditions (13) become:

$$a_1^2 - 2a_0 \cdot a_2 = -17{,}284 < 0;\ \ a_3^2 - a_2 \cdot a_4 = -192{,}23 < 0$$

$$a_5^2 - 2a_4 \cdot a_6 = -370{,}7 < 0$$

All the conditions (10) and (13) being accomplished, the equation (15) has complex roots with real part negative and they are not real negative that means the structure is stable and the characteristic equation (15) has the following roots:

$$\lambda_{1,2} = -0{,}00899 \pm 1{,}14376i$$
$$\lambda_{3,4} = -0{,}039231 \pm 1{,}6645i$$
$$\lambda_{5,6} = -0{,}08079 \pm 2{,}1434i$$

The damped eigenpulsations are given by (6)

$$p_1^a = 1{,}14376\,s^{-1};\quad p_2^a = 1{,}6645\,s^{-1};\quad p_3^a = 2{,}1434\,s^{-1}$$

Case b) If the damping factor $k_1 = 12$, then the characteristic equation (8)of the damped eigenpulsations is:

$$\lambda^6 + 3{,}3\,\lambda^5 + 10{,}5866\lambda^4 + 16{,}444\lambda^3 + 26{,}0533\lambda^2 + 16{,}9999\lambda + 16{,}(6) = 0 \qquad (18)$$

where:

$a_0= 1$; $a_1= 3{,}3$; $a_2= 10{,}5866$; $a_3= 16{,}444$; $a_4= 26{,}0533$; $a_5= 16{,}9999$; $a_6= 16{,}(6)=0$.

Hurwitz determinants attached to the polynomial (18) are:

det $H_1 = a_1= 3{,}3 > 0$; det $H_2 = 18{,}489 > 0$; det $H_3 = 76{,}462 > 0$;
det $H_4 = 853{,}48 >0$; det $H_5 = 864{,}136 >0$; det $H_6 = 14402{,}307 >0$.

All the coefficients of the characteristic equation are positive and the conditions (13) are accomplished:

$$a_1^2 - 2a_0 \cdot a_2 = -10{,}28 < 0\,;\ a_3^2 - a_2 \cdot a_4 = -5{,}41 < 0\,;\ a_5^2 - 2a_4 \cdot a_6 = -579{,}45 < 0$$

For the equation (18) the conditions (10) and (13) are accomplished, when the equation (18) has complex roots with the part negative, and they are not real negative.

$$\lambda_{1,2} = -0{,}10729 \pm 1{,}14274i$$
$$\lambda_{3,4} = -0{,}3226 \pm 1{,}627258i$$
$$\lambda_{5,6} = -1{,}31088 \pm 1{,}695479i$$

In this case the damped eigenpulsations are:

$$p_1^a = 1{,}14274\,s^{-1};\quad p_2^a = 1{,}627258\,s^{-1};\quad p_3^a = 1{,}695479\,s^{-1}.$$

One may notice from these two cases that, if factor for multiplying the coefficients of damping matrix **R** increases (the introduced damping increases), then the damped eigenpulsations decreases.

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