Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol. XII (1996),71-76

Dedicated to the 35[th] anniversary of the University of Baia Mare

ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO A CLASS OF HYPERBOLIC SYSTEMS

Rodica LUCA–TUDORACHE

Abstract In this paper we shall study the asymptotic behaviour of solutions to a class of nonlinear hyperbolic systems.

The purpose of the present paper is to investigate the asymptotic behaviour of the solutions of the problem:

$$
\text{(S)} \qquad \begin{cases} \ell(x)\dfrac{\partial i}{\partial t} + \dfrac{\partial v}{\partial x} + A(i) \ni f(t,x) \\[2ex] c(x)\dfrac{\partial v}{\partial t} + \dfrac{\partial i}{\partial x} + B(v) \ni g(t,x), \end{cases}
$$

$$0 < x < 1, \; t > 0$$

with the boundary condition:

$$
\text{(BC)} \qquad \begin{pmatrix} i(t,0) \\ -i(t,1) \\ S\left(\dfrac{dw}{dt}(t)\right) \end{pmatrix} \in -G \begin{pmatrix} v(t,0) \\ v(t,1) \\ w(t) \end{pmatrix}, \; t > 0
$$

and the initial data:

$$
\text{(IC)} \qquad \begin{cases} i(0,x) = i_0(x), \; v(0,x) = v_0(x), \quad 0 < x < 1 \\ w(0) = w_0. \end{cases}
$$

The existence, uniqueness and regularity properties of the solutions to the problem (S), (BC), (IC) have been investigated in [4] (see also [3]).

In all which follows we denote by (H1)–(H4) the following assumptions:

(H1) $\ell = diag(\ell_1, ..., \ell_n)$, $c = diag(c_1, ..., c_n)$ with $\ell_k, c_k \in L^\infty(0,1)$, $k = \overline{1,n}$ and $\ell_k(x) \geq k_0 > 0$, $c_k(x) \geq k_0 > 0$, for a.a. $x \in (0,1)$.

(H2) $A : D(A) \subset I\!R^n \longrightarrow I\!R^n$, $A = \partial\varphi$, where $\varphi : I\!R^n \longrightarrow (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function with $D(\varphi) \subsetneq I\!R^n$ and $B : D(B) = I\!R^n \longrightarrow I\!R^n$ is a maximal monotone operator (possibly multivalued).

Moreover: $(-Int\, D(\varphi)) \times (Int\, D(\varphi)) \times I\!R^m \cap R(G) \neq \emptyset$.

(H3) $G : D(G) \subset I\!R^{2n+m} \longrightarrow I\!R^{2n+m}$ is a maximal monotone operator (possibly multivalued), $D(G) \neq \emptyset$. Moreover, $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ with:

$$G_{11} : D(G_{11}) \subset I\!R^{2n} \longrightarrow I\!R^{2n}, \quad G_{12} : D(G_{12}) \subset I\!R^m \longrightarrow I\!R^{2n},$$
$$G_{21} : D(G_{21}) \subset I\!R^{2n} \longrightarrow I\!R^m, \quad G_{22} : D(G_{22}) \subset I\!R^m \longrightarrow I\!R^m.$$

(H4) $S = diag(s_1, ..., s_m)$ with $s_j > 0$, $j = \overline{1,m}$.

We consider the following *spaces* $X = (L^2(0,1; I\!R^n))^2$, $I\!R^m$ and $Y = X \times I\!R^m$ with the corresponding scalar products:

$$< f, g >_X = < f_1, g_1 >_{L^2(0,1;I\!R^n)} + < f_2, g_2 >_{L^2(0,1;I\!R^n)},$$
$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in X,$$
$$< x, y >_s = \sum_{i=1}^m s_i x_i y_i, \quad x, y \in I\!R^m$$

and

$$< \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} >_Y = < f, g >_X + < x, y >_s, \quad \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} \in Y.$$

We define the *operators* $C : D(C) \subset Y \longrightarrow Y$,

$$D(C) = \left\{ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y; \quad i, v \in H^1(0,1; I\!R^n); \right.$$
$$\left. \begin{pmatrix} \gamma_0 v \\ w \end{pmatrix} \in D(G), \ \gamma_1 i \in -G_{11}(\gamma_0 v) - G_{12}(w) \right\}$$

with $\gamma_1 i = \begin{pmatrix} i(0) \\ -i(1) \end{pmatrix}$, $\gamma_0 v = \begin{pmatrix} v(0) \\ v(1) \end{pmatrix}$,

$$C \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} v' \\ i' \\ S^{-1}G_{21}(\gamma_0 v) + S^{-1}G_{22}(w) \end{pmatrix}, \quad \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in D(C),$$

$$\tilde{A} : D(\tilde{A}) \subset C([0,1]; I\!R^n) \longrightarrow M(0,1; I\!R^n) \ \ (= (C([0,1]; I\!R^n))'),$$

$$\tilde{A}(p) = \left\{ \mu \in M(0,1; I\!R^n); \ \mu(p - q) \geq \int_0^1 \varphi(p(x))dx - \right.$$
$$\left. - \int_0^1 \varphi(q(x))dx, \quad \forall q \in C([0,1]; I\!R^n) \right\},$$

$$D(\tilde{A}) = \{ p \in C([0,1]; I\!R^n), \quad \tilde{A}(p) \neq \emptyset \} \ \text{ and}$$

$$\mathcal{A} : D(\mathcal{A}) \subset Y \longrightarrow Y,$$

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y;\ i \in H^1(0,1;\mathbb{R}^n);\ v \ni v_1 \in BV(0,1;\mathbb{R}^n);\ \exists \mu \in \tilde{A}(i) \right.$$
$$\left. \text{such that } \begin{pmatrix} \gamma_0 v_1 \\ w \end{pmatrix} \in D(G),\ \gamma_1 i \in -G_{11}(\gamma_0 v_1) - G_{12}(w),\ \mu + dv_1 \in L^2(0,1;\mathbb{R}^n) \right\}$$

$$\mathcal{A} \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \left\{ \begin{pmatrix} p \\ q \\ r \end{pmatrix};\ p \in (K_{ivw} + \tilde{A}(i)) \cap L^2(0,1;\mathbb{R}^n);\ q \in i' + \overline{B}(v), \right.$$
$$\left. r \in S^{-1}G_{21}(\gamma_0 v_1) + S^{-1}G_{22}(w) \right\},\ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in D(\mathcal{A}),$$

where $K_{ivw} = \{dv_1;\ v_1 \in BV(0,1;\mathbb{R}^n) \text{ from } D(\mathcal{A})\}$ and $\overline{B}$ is the canonical extension of B to $L^2(0,1;\mathbb{R}^n)$.

Lemma 1 *If* (H3) *and* (H4) *hold, then the operator C is maximal monotone.*

For the proof of Lemma 1 see [3].

Lemma 2 *If* (H2), (H3) *and* (H4) *hold, then the operator $\mathcal{A}$ is maximal monotone.*

For the proof of Lemma 2 see [3,4].

Theorem 1 a) *Suppose that the assumptions* (H1)–(H4) *hold. If $f, g \in W^{1,1}(0,T; L^2(0,1;\mathbb{R}^n))$ ($T > 0$ fixed) and $col(i_0, v_0, w_0) \in D(\mathcal{A})$, then the problem* (S), (BC), (IC) *has a unique strong solution $col(i,v,w) \in W^{1,\infty}(0,T;Y)$. Moreover $i, v \in L^\infty((0,T) \times (0,1);\mathbb{R}^n)$, $\partial i/\partial x \in L^\infty(0,T;L^2(0,1;\mathbb{R}^n))$.*

b) *Suppose that the assumptions* (H1)–(H4) *hold. If $f, g \in L^1(0,T;L^2(0,1;\mathbb{R}^n))$ ($T > 0$ fixed) and $col(i_0, v_0, w_0) \in \overline{D(\mathcal{A})}$ then the problem* (S), (BC), (IC) *has a unique weak solution $col(i,v,w) \in C([0,T];Y)$.*

For the proof of Theorem 1 see [3,4].

Theorem 2 *Suppose that the assumptions* (H1)–(H4) *hold and $f, g \in L^1(\mathbb{R}_+; ;L^2(0,1;\mathbb{R}^n))$, $col(i_0, v_0, w_0) \in \overline{D(\mathcal{A})}$. Moreover, we assume that $B = \partial\psi$, where $\psi : \mathbb{R}^n \longrightarrow (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function with $D(\psi) = \mathbb{R}^n$, $A^{-1}(0) = B^{-1}(0) = \{0\}$, $0 \in G(0)$ and*

(H5) *There exists $K > 0$ such that for all $x \in D(G)$, $x = (x^a, x^b) \in \mathbb{R}^{2n} \times \mathbb{R}^m$ and for all $w \in G(x)$:*

$$< w, x >_{\mathbb{R}^{2n+m}} \geq K \|x^b\|^2_{\mathbb{R}^m}.$$

Then:

$$i(t,\cdot) \longrightarrow 0, \text{ as } t \longrightarrow \infty, \text{ strongly in } L^2(0,1;\mathbb{R}^n) \tag{1}$$

$$v(t,\cdot) \longrightarrow 0, \text{ as } t \longrightarrow \infty, \text{ weakly in } L^2(0,1;\mathbb{R}^n) \tag{2}$$

$$w(t) \longrightarrow 0, \text{ as } t \longrightarrow \infty, \text{ in } \mathbb{R}^m, \tag{3}$$

where $col(i,v,w)$ is the weak solution of the problem (S), (BC), (IC) *corresponding to the data (f, g, i_0, v_0, w_0).*

If, in addition, $col(i_0, v_0, w_0) \in D(\mathcal{A})$, $f, g \in W^{1,1}(\mathbb{R}_+; L^2(0,1;\mathbb{R}^n))$ then:

$$i(t,\cdot) \longrightarrow 0, \text{ as } t \longrightarrow \infty, \text{ weakly in } H^1(0,1;\mathbb{R}^n). \tag{4}$$

Sketch of proof We assume, without loss of generality that G is single–valued and $\ell(x) = c(x) = I_n$, for a.a. $x \in (0,1)$. The general case can be treated using a similar argument as that we used in the proof of Theorem 1 (see [3]). First, we shall prove (1) and (2) in the weak topology of $L^2(0,1;\mathbb{R}^n)$ and (3). We suppose, for the moment, that $f = g \equiv 0$ and we shall show that the operator $\mathcal{A}$ is demipositive in comparison with 0 (see [5]). For, we consider the sequences $\{col(i_m, v_m, w_m)\}_m \subset D(\mathcal{A})$ and $\{col(p_m, q_m, r_m)\}$,

$$\begin{pmatrix} p_m \\ q_m \\ r_m \end{pmatrix} \in \mathcal{A} \begin{pmatrix} i_m \\ v_m \\ w_m \end{pmatrix} \tag{5}$$

with the properties $i_m \longrightarrow i$ and $v_m \longrightarrow v$, as $m \longrightarrow \infty$, weakly in $L^2(0,1;\mathbb{R}^n)$, $w_m \longrightarrow w$, as $m \longrightarrow \infty$ in $\mathbb{R}^m$, $\{col(p_m, q_m, r_m)\}_m$ is bounded in Y and

$$\lim_{m\to\infty} < \begin{pmatrix} p_m \\ q_m \\ r_m \end{pmatrix}, \begin{pmatrix} i_m \\ v_m \\ w_m \end{pmatrix} >_Y = 0. \tag{6}$$

In these conditions, we shall prove that $0 \in \mathcal{A}(col(i, v, w))$. Using (5) we deduce that there exist $v_{m,1} \in v_m$, $v_{m,1} \in BV(0,1;\mathbb{R}^n)$, $\mu_m \in \tilde{A}(i_m)$, $\delta_m \in \tilde{B}(v_m)$ such that:

$$\begin{cases} p_m = dv_{m,1} + \mu_m \\ q_m = i'_m + \delta_m, \text{ in } L^2(0,1;\mathbb{R}^n) \\ r_m = S^{-1}G_{21}(\gamma_0 v_{m,1}) + S^{-1}G_{22}(w_m) \\ \gamma_1 i_m = -G_{11}(\gamma_0 v_{m,1}) - G_{12}(w_m). \end{cases} \tag{7}$$

By the relations (6) and (7) we obtain:

$$\lim_{m\to\infty}\left[< \begin{pmatrix} \gamma_0 v_{m,1} \\ w_m \end{pmatrix}, G\begin{pmatrix} \gamma_0 v_{m,1} \\ w_m \end{pmatrix} >_{\mathbb{R}^{2n+m}} + \int_0^1 < \delta_m(x), v_m(x) >_{\mathbb{R}^n} dx + \mu_m(i_m)\right] = 0. \tag{8}$$

We suppose, without loss of generality, that $\varphi(0) = \psi(0) = 0$; then $\varphi \geq 0$ and $\psi \geq 0$. Using the properties of the operators G, A and B, by (8) we deduce:

$$\begin{aligned} &\lim_{m\to\infty} \int_0^1 \varphi(i_m(x))dx = \lim \int_0^1 \psi(v_m(x))dx = \\ &= \lim_{m\to\infty} < \begin{pmatrix} \gamma_0 v_{m,1} \\ w_m \end{pmatrix}, G\begin{pmatrix} \gamma_0 v_{m,1} \\ w_m \end{pmatrix} >_{\mathbb{R}^{2n+m}} = 0. \end{aligned} \tag{9}$$

So:

$$\lim_{m\to\infty} \Phi(i_m) = \lim_{m\to\infty} \Psi(v_m) = 0, \tag{10}$$

where $\Phi, \Psi : L^2(0,1;\mathbb{R}^n) \longrightarrow (-\infty, +\infty]$ are defined by:

$$\Phi(\xi) = \begin{cases} \int_0^1 \varphi(\xi(x))dx, & \text{if } \varphi \circ \xi \in L^1(0,1) \\ +\infty, & \text{in rest,} \end{cases}$$

$$\Psi(\xi) = \begin{cases} \int_0^1 \psi(\xi(x))dx, & \text{if } \psi \circ \xi \in L^1(0,1) \\ +\infty, & \text{in rest.} \end{cases}$$

Because these functions are weak lower semicontinuous and $i_m \to i$, $v_m \to v$, as $m \to \infty$, weakly in $L^2(0,1;\mathbb{R}^n)$, by (10) it follows that $\varphi(i(x)) = \psi(v(x)) = 0$, for a.a. $x \in (0,1)$. Using the assumption $A^{-1}(0) = B^{-1}(0) = \{0\}$ we deduce that $0 \in \mathbb{R}^n$ is the unique minimum point for the functions φ and ψ. Therefore $i(x) = v(x) = 0$, for a.a. $x \in (0,1)$.

Now, using (9), the assumption (H5) and $w_m \longrightarrow w$ as $m \longrightarrow \infty$, we deduce that $w = 0$. So, we obtain that $col(i, v, w) = col(0,0,0) \in \mathcal{A}^{-1}(0)$, therefore $\Omega(0) \subset \mathcal{A}^{-1}(0)$. Hence the operator $\mathcal{A}$ is demipositive in comparison with 0 and, moreover, $\mathcal{A}^{-1}(0) = \{0\}$. Using some qualitative results for the asymptotic behaviour of the solutions of the evolution equations (see [5]), we deduce the relations (1) and (2) in the weak topology of $L^2(0,1;\mathbb{R}^n)$ and the relation (3).

In what follows, we shall prove the relation (4). For, let $f, g \in W^{1,1}(\mathbb{R}_+; L^2(0,1;\mathbb{R}^n))$ and $col(i_0, v_0, w_0) \in D(\mathcal{A})$. We consider the following approximate problem:

$$\begin{cases} \dfrac{d^+}{dt}\begin{pmatrix} i_\lambda(t) \\ v_\lambda(t) \\ w_\lambda(t) \end{pmatrix} + \mathcal{A}^\lambda \begin{pmatrix} i_\lambda(t) \\ v_\lambda(t) \\ w_\lambda(t) \end{pmatrix} = \begin{pmatrix} f(t,\cdot) \\ g(t,\cdot) \\ 0 \end{pmatrix}, \ t \in \mathbb{R}_+ \\[2ex] \begin{pmatrix} i_\lambda(0) \\ v_\lambda(0) \\ w_\lambda(0) \end{pmatrix} = (I + \mathcal{A}^\lambda)^{-1}\left(\begin{pmatrix} i_0 \\ v_0 \\ w_0 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right), \ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathcal{A}\begin{pmatrix} i_0 \\ v_0 \\ w_0 \end{pmatrix}, \\ \qquad \lambda > 0. \end{cases} \tag{11}$$

where $\mathcal{A}^\lambda : D(\mathcal{A}^\lambda) = D(C) \subset Y \longrightarrow Y$, $\mathcal{A}^\lambda = C + \mathcal{L}_\lambda$ and $\mathcal{L}_\lambda : Y \longrightarrow Y$ is defined by:

$$\mathcal{L}_\lambda \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} \bar{A}_\lambda(i) \\ \bar{B}_\lambda(v) \\ 0 \end{pmatrix}, \quad \forall \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y,$$

($\bar{A}_\lambda$ and $\bar{B}_\lambda$ are the canonical extensions to $L^2(0,1;\mathbb{R}^n)$ of the Yosida approximates A_λ and B_λ, respectively, of A and B).

Because $\mathcal{A}^\lambda(0) = 0$, $\forall \lambda > 0$, by (11) we deduce:

$$\frac{1}{2}\left\| \begin{pmatrix} i_\lambda(t) \\ v_\lambda(t) \\ w_\lambda(t) \end{pmatrix} \right\|_Y^2 \le const. + \int_0^t \left\| \begin{pmatrix} f(s,\cdot) \\ g(s,\cdot) \end{pmatrix} \right\|_X \left\| \begin{pmatrix} i_\lambda(s) \\ v_\lambda(s) \\ w_\lambda(s) \end{pmatrix} \right\|_Y ds, \tag{12}$$

$$t \in \mathbb{R}_+, \ \lambda > 0.$$

Using the Gronwall's Lemma and $f, g \in W^{1,1}(\mathbb{R}_+; L^2(0,1;\mathbb{R}^n))$, by (12) we obtain:

$$\sup\{\|i_\lambda(t,\cdot)\|_{L^2(0,1;\mathbb{R}^n)} + \|v_\lambda(t,\cdot)\|_{L^2(0,1;\mathbb{R}^n)} + \|w_\lambda(t)\|_{\mathbb{R}^m};\ t \ge 0,\ \lambda > 0\} < \infty.$$

Using a similar argument as that we used in the proof of Lemma 2 (see [4]) we get:

$$\sup\left\{\left\|\frac{\partial i_\lambda}{\partial x}(t,\cdot)\right\|_{L^2(0,1;\mathbb{R}^n)};\ t \ge 0,\ \lambda > 0\right\} < \infty. \tag{13}$$

Because $i_\lambda \longrightarrow i$, as $\lambda \longrightarrow 0$, in $C([0,T]; L^2(0,1;\mathbb{R}^n))$, $\forall T > 0$, by (13) we have:

$$\sup\left\{\left\|\frac{\partial i}{\partial x}(t,\cdot)\right\|_{L^2(0,1;\mathbb{R}^n)};\ t \ge 0\right\} < \infty. \tag{14}$$

Because $\mathcal{A}^{-1}(0) = \{0\}$, by the above relation we deduce that the set $\{i(t,\cdot), t \in \mathbb{R}_+\}$ is bounded in $H^1(0,1;\mathbb{R}^n)$. This last conclusion combined by (1) in the weak topology give us the relation (4).

To conclude the proof of Theorem we must show that if $col(i_0, v_0, w_0) \in \overline{D(\mathcal{A})}$, $f, g \in L^1(\mathbb{R}_+; L^2(0,1;\mathbb{R}^n))$, then $i(t,\cdot)\longrightarrow 0$, as $t\longrightarrow\infty$, strongly in $L^2(0,1;\mathbb{R}^n)$. For, let $\{col(i_0^m, v_0^m, w_0^m)\}_m \subset D(\mathcal{A})$ be such that $col(i_0^m, v_0^m, w_0^m)\longrightarrow col(i_0, v_0, w_0)$, as $m\longrightarrow\infty$, in Y and let $\{col(f_m, g_m)\}_m \subset W^{1,1}(\mathbb{R}_+; X)$ be such that $col(f_m, g_m)\longrightarrow col(f, g)$, as $m\longrightarrow\infty$ in X. We denote by $col(i_m, v_m, w_m)$ the strong solution of the problem (S), (BC), (IC) corresponding to the data $(f_m, g_m, i_0^m, v_0^m, w_0^m)$. Then:

$$\|i(t,\cdot)\|_{L^2(0,1;\mathbb{R}^n)} \le \left\| \begin{pmatrix} i_0^m \\ v_0^m \\ w_0^m \end{pmatrix} - \begin{pmatrix} i_0 \\ v_0 \\ w_0 \end{pmatrix} \right\|_Y + \int_0^t \left\| \begin{pmatrix} f_m(s,\cdot) \\ g_m(s,\cdot) \end{pmatrix} - \begin{pmatrix} f(s,\cdot) \\ g(s,\cdot) \end{pmatrix} \right\|_X ds + \tag{15}$$

$$+\|i_m(t,\cdot)\|_{L^2(0,1;\mathbb{R}^n)}, \ t \in \mathbb{R}_+, \ \forall m \in \mathbb{N}.$$

Because, for each m, $i_m(t,\cdot)\longrightarrow 0$, as $t\longrightarrow\infty$, weakly in $H^1(0,1;\mathbb{R}^n)$ (see (4)) and, also, strongly in $L^2(0,1;\mathbb{R}^n)$, by (15) we deduce:

$$\limsup_{t\longrightarrow\infty} \|i(t,\cdot)\|_{L^2(0,1;\mathbb{R}^n)} \le \left\| \begin{pmatrix} i_0^m \\ v_0^m \\ w_0^m \end{pmatrix} - \begin{pmatrix} i_0 \\ v_0 \\ w_0 \end{pmatrix} \right\|_Y + \left\| \begin{pmatrix} f_m - f \\ g_m - g \end{pmatrix} \right\|_{L^1(\mathbb{R}_+;X)}, \forall m \in \mathbb{N}.$$

This last inequality give us that $i(t,\cdot)\longrightarrow 0$, as $t\longrightarrow\infty$, strongly in $L^2(0,1;\mathbb{R}^n)$, thereby proving the theorem.

q.e.d.

REFERENCES

1. V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces,* Noordhoff, Leyden, 1976.

2. R. Luca, *An existence result for a nonlinear hyperbolic system,* Diff. Integral Equations, 8(1995), 887–900.

3. R. Luca, *Probleme la limită pentru sisteme cu derivate parţiale de tip hiperbolic şi aplicaţii,* Doctoral Thesis, Department of Mathematics, University of Iaşi, 1995.

4. R. Luca–Tudorache, *Mixed problems for a class of nonlinear hyperbolic systems,* to appear.

5. G.Moroşanu, *Nonlinear Evolution Equations and Applications,* D.Reidel, Dordrecht, 1988.

Received 01.06.1996

Technical University of Iaşi
Department of Mathematics
RO-6600 Iaşi
ROMANIA