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MIXED PROBLEMS FOR A CLASS OF NONLINEAR HYPERBOLIC SYSTEMS

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Abstract In this paper we shall study the qualitative properties (existence, uniqueness and regularity) of solutions to a class of nonlinear hyperbolic systems.

The purpose of the present paper is to investigate the following problem:

$$\text{(S)} \qquad \begin{cases} \ell(x)\dfrac{\partial i}{\partial t} + \dfrac{\partial v}{\partial x} + A(i) \ni f(t,x) \\[2ex] c(x)\dfrac{\partial v}{\partial t} + \dfrac{\partial i}{\partial x} + B(v) \ni g(t,x), \end{cases}$$

$$0 < x < 1, \; 0 < t < T$$

with the boundary condition:

$$\text{(BC)} \qquad \begin{pmatrix} i(t,0) \\ -i(t,1) \\ S\left(\dfrac{dw}{dt}(t)\right) \end{pmatrix} \in -G \begin{pmatrix} v(t,0) \\ v(t,1) \\ w(t) \end{pmatrix}, \; 0 < t < T$$

and the initial data:

$$\text{(IC)} \qquad \begin{cases} i(0,x) = i_0(x), \; v(0,x) = v_0(x), \quad 0 < x < 1 \\ w(0) = w_0. \end{cases}$$

This problem is an extension of the problem studied in [7], where no function $w(t)$ appears. Such problems have applications in electronics and hydraulics. Others nonlinear hyperbolic systems with boundary conditions of this type and different examples have been investigated in [3-8].

We introduce the assumptions that we shall use in the sequel:

(H1) $\ell = diag(\ell_1, ..., \ell_n)$, $c = diag(c_1, ..., c_n)$ with $\ell_k, c_k \in L^\infty(0,1)$, $k = \overline{1,n}$ and $\ell_k(x) \geq k_0 > 0$, $c_k(x) \geq k_0 > 0$, for a.a $x \in (0,1)$.

(H2) $A : D(A) \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $A = \partial\varphi$, where $\varphi : \mathbb{R}^n \longrightarrow (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function with $D(\varphi) \subsetneq \mathbb{R}^n$ and $B : D(B) = \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a maximal monotone operator (possibly multivalued).

Moreover: $(-Int\, D(\varphi)) \times (Int\, D(\varphi)) \times \mathbb{R}^m \cap R(G) \neq \emptyset$.

(H3) $G : D(G) \subset \mathbb{R}^{2n+m} \longrightarrow \mathbb{R}^{2n+m}$ is a maximal monotone operator (possibly multivalued), $D(G) \neq \emptyset$. Moreover, $G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$ with:

$$G_{11} : D(G_{11}) \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}, \quad G_{12} : D(G_{12}) \subset \mathbb{R}^m \longrightarrow \mathbb{R}^{2n},$$
$$G_{21} : D(G_{21}) \subset \mathbb{R}^{2n} \longrightarrow \mathbb{R}^m, \quad G_{22} : D(G_{22}) \subset \mathbb{R}^m \longrightarrow \mathbb{R}^m.$$

(H4) $S = diag(s_1, ..., s_m)$ with $s_j > 0$, $j = \overline{1,m}$.

We consider the following *spaces* $X = (L^2(0,1;\mathbb{R}^n))^2$, $\mathbb{R}^m$ and $Y = X \times \mathbb{R}^m$ with the corresponding scalar products:

$$< f, g >_X = < f_1, g_1 >_{L^2(0,1;\mathbb{R}^n)} + < f_2, g_2 >_{L^2(0,1;\mathbb{R}^n)} =$$

$$= \sum_{i=1}^n \int_0^1 f_{1i}(x) g_{1i}(x) dx + \sum_{i=1}^n \int_0^1 f_{2i}(x) g_{2i}(x) dx,$$

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in X, \quad f_1 = col(f_{11}, ..., f_{1n}), ...,$$

$$< x, y >_s = \sum_{i=1}^m s_i x_i y_i, \quad x, y \in \mathbb{R}^m$$

and

$$< \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} >_Y = < f, g >_X + < x, y >_s, \quad \begin{pmatrix} f \\ x \end{pmatrix}, \begin{pmatrix} g \\ y \end{pmatrix} \in Y.$$

We define the *operator*: $C : D(C) \subset Y \longrightarrow Y$,

$$D(C) = \left\{ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y; \quad i, v \in H^1(0,1;\mathbb{R}^n); \right.$$
$$\left. \begin{pmatrix} \gamma_0 v \\ w \end{pmatrix} \in D(G), \; \gamma_1 i \in -G_{11}(\gamma_0 v) - G_{12}(w) \right\}$$

with $\gamma_1 i = \begin{pmatrix} i(0) \\ -i(1) \end{pmatrix}$, $\gamma_0 v = \begin{pmatrix} v(0) \\ v(1) \end{pmatrix}$,

$$C \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} v' \\ i' \\ S^{-1} G_{21}(\gamma_0 v) + S^{-1} G_{22}(w) \end{pmatrix}, \quad \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in D(C).$$

Lemma 1 *If* (H3) *and* (H4) *hold, then the operator* C *is maximal monotone.*

For the proof of Lemma 1 see [5] (see also [4] for a different proof).

In what follows we define the following *operators*:

$$\tilde{A} : D(\tilde{A}) \subset C([0,1];\mathbb{R}^n) \longrightarrow M(0,1;\mathbb{R}^n) \ \ (= (C([0,1];\mathbb{R}^n))'),$$

$$\tilde{A}(p) = \Big\{\mu \in M(0,1;\mathbb{R}^n);\ \mu(p-q) \geq \int_0^1 \varphi(p(x))dx - \\ - \int_0^1 \varphi(q(x))dx,\ \ \forall q \in C([0,1];\mathbb{R}^n)\Big\},$$

$$D(\tilde{A}) = \{p \in C([0,1];\mathbb{R}^n),\ \ \tilde{A}(p) \neq \emptyset\} \ \text{ and}$$

$\mathcal{A} : D(\mathcal{A}) \subset Y \longrightarrow Y$,

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y;\ i \in H^1(0,1;\mathbb{R}^n);\ v \ni v_1 \in BV(0,1;\mathbb{R}^n);\ \exists \mu \in \tilde{A}(i) \right.$$
$$\left. \text{such that } \begin{pmatrix} \gamma_0 v_1 \\ w \end{pmatrix} \in D(G),\ \gamma_1 i \in -G_{11}(\gamma_0 v_1) - G_{12}(w),\ \mu + dv_1 \in L^2(0,1;\mathbb{R}^n) \right\}$$

$$\mathcal{A}\begin{pmatrix} i \\ v \\ w \end{pmatrix} = \left\{ \begin{pmatrix} p \\ q \\ r \end{pmatrix};\ p \in (K_{ivw} + \tilde{A}(i)) \cap L^2(0,1;\mathbb{R}^n);\ q \in i' + \overline{B}(v), \right.$$
$$\left. r \in S^{-1}G_{21}(\gamma_0 v_1) + S^{-1}G_{22}(w) \right\},\ \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in D(\mathcal{A}),$$

where $K_{ivw} = \{dv_1;\ v_1 \in BV(0,1;\mathbb{R}^n) \text{ from } D(\mathcal{A})\}$, and $\overline{B}$ is the canonical extension of B to $L^2(0,1;\mathbb{R}^n)$. We understand by $\mu + dv_1 \in L^2(0,1;\mathbb{R}^n)$ that the measure $\mu + dv_1 : C([0,1];\mathbb{R}^n) \longrightarrow \mathbb{R}$ can be extended to a functional on $L^2(0,1;\mathbb{R}^n)$ and then this is identified with the corresponding function from $L^2(0,1;\mathbb{R}^n)$.

Lemma2 *If* (H2), (H3) *and* (H4) *hold, then the operator* $\mathcal{A}$ *is maximal monotone.*

Sketch of proof We shall consider without loss of generality that G is a single-valued operator. Using the assumption (H2) we deduce that there exists $col(p^*, q^*, r^*) \in D(\mathcal{A})$ such that the set $M = \{p^*(x);\ 0 \leq x \leq 1\}$ is compact and included in $Int\, D(\varphi) = Int\, D(A)$. Because A is monotone the set $A(M)$ is bounded. It is easy to check that $\mathcal{A}$ is monotone. To show that $\mathcal{A}$ is maximal monotone it is sufficient to prove that for any fixed $col(p,q,r) \in Y$ there exists $col(i,v,w) \in D(\mathcal{A})$ such that:

$$(I + \mathcal{A})\begin{pmatrix} i \\ v \\ w \end{pmatrix} \ni \begin{pmatrix} p \\ q \\ r \end{pmatrix} \tag{1}$$

For, let $col(p,q,r) \in Y$ be arbitrary, but fixed. The equation (1) can be equivalently written as:

$$\begin{cases} i + K_{ivw} + \tilde{A}(i) \ni p \\ v + i' + \overline{B}(v) \ni q, \quad \text{in } L^2(0,1;\mathbb{R}^n) \\ w + S^{-1}G_{21}(\gamma_0 v_1) + S^{-1}G_{22}(w) = r, \quad \text{in } \mathbb{R}^m \\ \gamma_1 i = -G_{11}(\gamma_0 v_1) - G_{12}(w), \\ \text{with } v_1 \in v,\ v_1 \in BV(0,1;\mathbb{R}^n),\ \exists \mu \in \tilde{A}(i) \text{ such that } \mu + dv_1 \in L^2(0,1;\mathbb{R}^n). \end{cases} \tag{2}$$

We define the *operator* $\mathcal{L}_\lambda : Y \longrightarrow Y$, $\lambda > 0$ by:

$$\mathcal{L}_\lambda \begin{pmatrix} i \\ v \\ w \end{pmatrix} = \begin{pmatrix} \bar{A}_\lambda(i) \\ \bar{B}_\lambda(v) \\ 0 \end{pmatrix}, \quad \forall \begin{pmatrix} i \\ v \\ w \end{pmatrix} \in Y,$$

where $\bar{A}_\lambda$ and $\bar{B}_\lambda$ are the canonical extensions to $L^2(0,1;\mathbb{R}^n)$ of the Yosida approximates A_λ and B_λ, respectively, of A and B. For each $\lambda > 0$, the operator $\mathcal{L}_\lambda$ is maximal monotone with $D(\mathcal{L}_\lambda) = Y$. Using the well-known Rockafellar's theorem (see [1]) we deduce that the operator $C + \mathcal{L}_\lambda : D(C) \subset Y \longrightarrow Y$ is maximal monotone, $\lambda > 0$. Therefore, for each $\lambda > 0$ there exists $col(i_\lambda, v_\lambda, w_\lambda) \in D(C)$ such that:

$$\begin{pmatrix} i_\lambda \\ v_\lambda \\ w_\lambda \end{pmatrix} + C \begin{pmatrix} i_\lambda \\ v_\lambda \\ w_\lambda \end{pmatrix} + \mathcal{L}_\lambda \begin{pmatrix} i_\lambda \\ v_\lambda \\ w_\lambda \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \text{in } Y. \tag{3}$$

For a fixed element $col(i_0, v_0, w_0) \in D(C)$, using a standard argument, we can prove that the sequences:

$$\begin{cases} \{i_\lambda;\ \lambda > 0\}, \{v_\lambda;\ \lambda > 0\} \text{ are bounded in } L^2(0,1;\mathbb{R}^n), \\ \{w_\lambda;\ \lambda > 0\} \text{ is bounded in } \mathbb{R}^m. \end{cases} \tag{4}$$

The equation (3) is equivalent to:

$$\begin{cases} i_\lambda + v'_\lambda + \bar{A}_\lambda(i_\lambda) = p \\ v_\lambda + i'_\lambda + \bar{B}_\lambda(v_\lambda) = q, \quad \text{in } L^2(0,1;\mathbb{R}^n) \\ w_\lambda + S^{-1}G_{21}(\gamma_0 v_\lambda) + S^{-1}G_{22}(w_\lambda) = r, \text{ in } \mathbb{R}^m \\ \begin{pmatrix} \gamma_0 v_\lambda \\ w_\lambda \end{pmatrix} \in D(G), \ \gamma_1 i_\lambda = -G_{11}(\gamma_0 v_\lambda) - G_{12}(w_\lambda). \end{cases} \tag{5}$$

By some computations involving the set M (see [4] for more details) we can show that the sequences:

$$\{i'_\lambda;\ \lambda > 0\} \text{ and } \{v'_\lambda;\ \lambda > 0\} \text{ are bounded in } L^1(0,1;\mathbb{R}^n). \tag{6}$$

Using the Sobolev's theorem, by (4) and (6) we deduce that the sequences $\{i_\lambda;\ \lambda > 0\}$ and $\{v_\lambda;\ \lambda > 0\}$ are bounded in $C([0,1];\mathbb{R}^n)$ and by $(5)_2$ that the sequence $\{i'_\lambda;\ \lambda > 0\}$ is bounded in $L^2(0,1;\mathbb{R}^n)$. By the Arzelà–Ascoli Criterion we have (eventually on subsequence):

$$i_\lambda \longrightarrow i, \text{ as } \lambda \longrightarrow 0, \text{ in } C([0,1];\mathbb{R}^n).$$

For v_λ we use the Helly's principle (see [9]) and we deduce the existence of $v_1 \in BV(0,1;\mathbb{R}^n)$ such that:

$$v_\lambda(x) \longrightarrow v_1(x), \text{ as } \lambda \longrightarrow 0, \ \forall x \in [0,1].$$

Next, by Lebesgue's Dominated Convergence Theorem, it follows that:

$$v_\lambda \longrightarrow v, \text{ as } \lambda \longrightarrow 0, \text{ strongly in } L^p(0,1;\mathbb{R}^n), \ 1 \le p \le \infty,$$

where v is the equivalence class of v_1.

So, using the above relations, we obtain, as $\lambda \longrightarrow 0$ (eventually on subsequences):

$$i'_\lambda \longrightarrow i', \text{ weakly in } L^2(0,1;\mathbb{R}^n)$$

$$v'_\lambda \longrightarrow dv_1, \text{ star–weakly in } M(0,1;\mathbb{R}^n)$$

$$\bar{A}_\lambda(i_\lambda) \longrightarrow \mu, \text{ star–weakly in } M(0,1;\mathbb{R}^n)$$

where $\mu = p - i - dv_1$,

$$\bar{B}_\lambda(v_\lambda) \longrightarrow \bar{B}(v), \text{ weakly in } L^2(0,1;\mathbb{R}^n)$$

$$w_\lambda \longrightarrow w, \text{ in } \mathbb{R}^m.$$

An easy computation shows that $\mu \in \tilde{A}(i)$. Since G is closed, letting $\lambda \longrightarrow 0$ in (5) we deduce (2), that is $col(i,v,w) \in D(\mathcal{A})$ is a solution of the equation (1).

q.e.d.

Lemma 3 *If $A = \partial\varphi$, where $\varphi : \mathbb{R}^n \longrightarrow (-\infty, +\infty]$ is a proper, convex and lower semicontinuous function, then for $\forall \mu \in \tilde{A}(p)$, we have:*

$$\begin{array}{ll} \mu_a(x) \in A(p(x)), & a.e.\ x \in (0,1), \\ \mu_s(p - w) \geq 0, & \forall w \in C([0,1];\mathbb{R}^n),\ w(x) \in \overline{D(\varphi)}, \end{array}$$

$\forall x \in [0,1]$, *(we denote by μ_a and μ_s the absolutely continuous part and the singular part, respectively, of the measure μ).*

For the proof of Lemma 3 see [10].

Theorem *Suppose that the assumptions* (H1)–(H4) *hold. If $f, g \in W^{1,1}(0,T;L^2(0,1;$ $;\mathbb{R}^n))$ and $col(i_0, v_0, w_0) \in D(\mathcal{A})$, then there exists an unique element $col(i,v,w)$ such that:*

(i) $i, v \in W^{1,\infty}(0,T;L^2(0,1;\mathbb{R}^n))$, $w \in W^{1,\infty}(0,T;\mathbb{R}^m)$.

(ii) $\forall t \in [0,T]$, $\begin{pmatrix} i(t,\cdot) \\ v(t,\cdot) \\ w(t) \end{pmatrix} \in D(\mathcal{A})$.

(iii) *For $\forall t \in [0,T]$, $v(t,\cdot) \in BV(0,1;\mathbb{R}^n)$ and for $\forall t \in [0,T)$, $col(i,v,w)$ satisfies the system* (S), *a.e. $x \in (0,1)$ and the boundary condition* (BC), *where $\partial i/\partial t$, $\partial v/\partial t$, and dw/dt are replaced by $\partial^+ i/\partial t$, $\partial^+ v/\partial t$, $d^+ w/dt$, respectively.*

(iv) *i, v, w satisfy the initial data* (IC).

Moreover:

$$\begin{array}{l} i, v \in L^\infty((0,T) \times (0,1);\mathbb{R}^n), \\ \partial i/\partial x \in L^\infty(0,T;L^2(0,1;\mathbb{R}^n)), \end{array} \tag{7}$$

and for $\forall t \in [0,T]$ the singular part of $v(t,\cdot)$ satisfies:

$$dv_s(t,\cdot)(i(t,\cdot) - \xi) \leq 0,\ \forall \xi \in C([0,1];\mathbb{R}^n) \text{ with } \xi(x) \in \overline{D(\varphi)},\ 0 \leq x \leq 1,$$

(we have identified the class $v(t,\cdot)$ with a representative, which together with $i(t,\cdot)$ and $w(t)$ satisfy (BC)*).*

Sketch of proof We suppose again that G is single–valued. First, we assume that $\ell(x) = c(x) = I_n$, for a.a. $x \in (0,1)$.

We consider the following Cauchy problem in the space Y:

$$\text{(P)} \qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix} i(t) \\ v(t) \\ w(t) \end{pmatrix} + \mathcal{A}\begin{pmatrix} i(t) \\ v(t) \\ w(t) \end{pmatrix} \ni \begin{pmatrix} f(t,\cdot) \\ g(t,\cdot) \\ 0 \end{pmatrix}, \quad 0 < t < T \\ \\ \begin{pmatrix} i(0) \\ v(0) \\ w(0) \end{pmatrix} = \begin{pmatrix} i^0 \\ v^0 \\ w^0 \end{pmatrix}. \end{cases}$$

By Lemma 2 the operator $\mathcal{A}$ is maximal monotone. Then, by the general theory of the evolution equations in Hilbert spaces, we deduce that the problem (P) has a unique strong solution $col(i,v,w) \in W^{1,\infty}(0,T;Y)$. Moreover, for each $t \in [0,T]$, $col(i(t),v(t),w(t)) \in$ $\in D(\mathcal{A})$ and the equation $(\text{P})_1$ is satisfied with d^+/dt instead of d/dt for any $t \in [0,T)$.

By the definition of the operator $\mathcal{A}$ it follows that for $\forall t \in [0,T]$, $i(t,\cdot) \in H^1(0,1;\mathbb{R}^n)$, $v(t,\cdot)$ contains a function $v^1(t,\cdot) \in BV(0,1;\mathbb{R}^n)$, such that $i(t,\cdot), v^1(t,\cdot)$ and $w(t)$ satisfy (BC). Besides, there exists a measure $\mu(t) \in \tilde{A}(i(t,\cdot))$ such that $\mu(t) + d_x v^1(t,\cdot) \in$ $\in L^2(0,1;\mathbb{R}^n)$ and for each $t \in [0,T)$ we have:

$$\begin{cases} \dfrac{\partial^+ i}{\partial t}(t,\cdot) + d_x v^1(t,\cdot) + \mu(t) = f(t,\cdot) \\ \dfrac{\partial^+ v}{\partial t}(t,\cdot) + \dfrac{\partial i}{\partial x}(t,\cdot) + B(v(t,\cdot)) \ni g(t,\cdot), \text{ in } L^2(0,1;\mathbb{R}^n) \\ S\dfrac{d^+ w}{dt}(t) + G_{21}((\gamma_0 v^1)(t)) + G_{22}(w(t)) = 0, \\ (\gamma_1 i)(t) = -G_{11}((\gamma_0 v^1)(t)) - G_{12}(w(t)). \end{cases}$$

We denote by $d_x v^1(t,\cdot)$ the measure generated by $v^1(t,\cdot) \in BV(0,1;\mathbb{R}^n)$.

Because $\mu(t) + d_x v^1(t,\cdot) \in L^2(0,1;\mathbb{R}^n)$ it follows that the singular part of this measure is 0, so:

$$\mu(t)_s = -d_x v^1_s(t,\cdot), \ \forall t \in [0,T],$$

where v^1_s is the singular part of the function v^1. By Lemma 3, we obtain:

$$d_x v^1_s(t,\cdot)(i(t,\cdot) - \xi) \le 0, \quad \forall \xi \in C([0,1];\mathbb{R}^n), \ \xi(x) \in \overline{D(\varphi)}, \ \forall x \in [0,1].$$

To prove (7) we consider the following approximate problem:

$$(\text{P})_\lambda \qquad \begin{cases} \dfrac{d}{dt}\begin{pmatrix} i(t) \\ v(t) \\ w(t) \end{pmatrix} + \mathcal{A}^\lambda\begin{pmatrix} i(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} f(t,\cdot) \\ g(t,\cdot) \\ 0 \end{pmatrix}, \ 0 < t < T, \text{ in } Y \\ \\ \begin{pmatrix} i(0) \\ v(0) \\ w(0) \end{pmatrix} = (I + \mathcal{A}^\lambda)^{-1}\left(\begin{pmatrix} i_0 \\ v_0 \\ w_0 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\right), \ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in \mathcal{A}\begin{pmatrix} i_0 \\ v_0 \\ w_0 \end{pmatrix}, \end{cases}$$

where $\mathcal{A}^\lambda : D(\mathcal{A}^\lambda) = D(C) \subset Y \longrightarrow Y$, $\mathcal{A}^\lambda = C + \mathcal{L}_\lambda$, $\lambda > 0$.

Adapting the proof of Lemma 2 , we deduce that:

$$(I+\delta\mathcal{A}^{\lambda})^{-1}\begin{pmatrix}p\\q\\r\end{pmatrix}\longrightarrow(I+\delta\mathcal{A})^{-1}\begin{pmatrix}p\\q\\r\end{pmatrix},\ \lambda\longrightarrow 0,\ \text{strongly in } Y, \tag{8}$$

for any $col(p,q,r)\in Y$, $\forall\delta>0$.

For $\lambda>0$ we denote by $col(i_\lambda,v_\lambda,w_\lambda)$ the solution of the problem $(\mathrm{P})_\lambda$. Using (8) we deduce:

$$\begin{pmatrix}i_\lambda(0)\\v_\lambda(0)\\w_\lambda(0)\end{pmatrix}\longrightarrow\begin{pmatrix}i_0\\v_0\\w_0\end{pmatrix},\ \lambda\longrightarrow 0,\ \text{strogly in } Y,$$

$$\mathcal{A}^{\lambda}\begin{pmatrix}i_\lambda(0)\\v_\lambda(0)\\w_\lambda(0)\end{pmatrix}\longrightarrow\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix},\ \lambda\longrightarrow 0,\ \text{strongly in } Y.$$

From the general theory of the evolution equations, we have:

$$\left\|\frac{d^+}{dt}\begin{pmatrix}i_\lambda(t)\\v_\lambda(t)\\w_\lambda(t)\end{pmatrix}\right\|_Y\leq\left\|\mathcal{A}^{\lambda}\begin{pmatrix}i_\lambda(0)\\v_\lambda(0)\\w_\lambda(0)\end{pmatrix}-\begin{pmatrix}f(0,\cdot)\\g(0,\cdot)\\0\end{pmatrix}\right\|_Y+$$

$$+\int_0^t\left\|\begin{pmatrix}\dfrac{\partial f}{\partial s}(s,\cdot)\\\dfrac{\partial g}{\partial s}(s,\cdot)\end{pmatrix}\right\|_X,\quad 0\leq t<T,\ \lambda>0.$$

By this last inequality, using also the assumptions of the theorem, we obtain:

$$\sup\left\{\left\|\frac{\partial^+ i_\lambda}{\partial t}(t,\cdot)\right\|_{L^2(0,1;\mathbb{R}^n)}+\left\|\frac{\partial^+ v_\lambda}{\partial t}(t,\cdot)\right\|_{L^2(0,1;\mathbb{R}^n)}+\right. \tag{9}$$

$$\left.+\left\|\frac{d^+ w_\lambda}{dt}(t)\right\|_{\mathbb{R}^m};\ 0\leq t<T,\ \lambda>0\right\}<\infty.$$

Using Brézis and Pazy's theorem (see [2]) we get:

$$\begin{aligned}&i_\lambda\longrightarrow i,\ v_\lambda\longrightarrow v,\ \text{as } \lambda\longrightarrow 0,\ \text{in } C([0,T];L^2(0,1;\mathbb{R}^n)),\\&w_\lambda\longrightarrow w,\ \text{as } \lambda\longrightarrow 0,\ \text{in } C([0,T];\mathbb{R}^m),\end{aligned} \tag{10}$$

where $col(i,v,w)$ is the solution of the problem (P).

Because $f,g\in W^{1,1}(0,T;L^2(0,1;\mathbb{R}^n))$, by $(\mathrm{P})_\lambda$ and (9) we have:

$$\sup\left\{\|\alpha_\lambda(t,\cdot)\|_{L^2(0,1;\mathbb{R}^n)}+\|\beta_\lambda(t,\cdot)\|_{L^2(0,1;\mathbb{R}^n)};\ 0\leq t<T,\ \lambda>0\right\}<\infty, \tag{11}$$

where $\alpha_\lambda=\partial v_\lambda/\partial x+A_\lambda(i_\lambda)$, $\beta_\lambda=\partial i_\lambda/\partial x+B_\lambda(v_\lambda)$.

The functions A_λ and B_λ being Lipschitz, by (11) we deduce that the distributions $\partial i_\lambda/\partial x$ and $\partial v_\lambda/\partial x\in L^\infty(0,T;L^2(0,1;\mathbb{R}^n))$. Now, using a similar argument as that we used in the proof of Lemma 2 (see [4]) we obtain:

$$\sup\left\{\left\|\frac{\partial i_\lambda}{\partial x}(t,\cdot)\right\|_{L^1(0,1;\mathbb{R}^n)}+\left\|\frac{\partial v_\lambda}{\partial x}(t,\cdot)\right\|_{L^1(0,1;\mathbb{R}^n)};\ 0\leq t<T,\ \lambda>0\right\}<\infty. \tag{12}$$

Using the Sobolev's theorem, the relations (10) and (12), we have:

$$\|i_\lambda(t,x)\|_{\mathbb{R}^n} + \|v_\lambda(t,x)\|_{\mathbb{R}^n} \le const., \quad \forall t \in [0,T], \ x \in [0,1], \ \lambda > 0. \qquad (13)$$

Now, by (11), (13) and $(P)_{\lambda,1}$ we deduce that:

$$\sup\left\{\left\|\frac{\partial i_\lambda}{\partial x}(t,\cdot)\right\|_{L^2(0,1;\mathbb{R}^n)} ; \ 0 \le t < T, \ \lambda > 0\right\} < \infty. \qquad (14)$$

So, using (10), (13) and (14) we deduce the regularity properties (7). For the general case $\ell(x)$, $c(x)$ we use a similar argument as that used in [8] (see for more details [4]).

q.e.d.

Remark If the assumptions (H1)–(H4) hold and $f, g \in L^1(0,T; L^2(0,1;\mathbb{R}^n))$, $col(i_0, v_0, w_0) \in \overline{D(\mathcal{A})}$, then the problem (P) has a unique weak solution $col(i,v,w) \in$ $\in C([0,T];Y)$. We say that this element $col(i,v,w)$ is a weak solution of the problem (S), (BC), (IC). This solution does not satisfie the boundary condition (BC), because $col(i(t,\cdot),$
$v(t,\cdot), w(t))$ does not generally belong to $D(\mathcal{A})$.

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