Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol. XII (1996),85-90

Dedicated to the 35th anniversary of the University of Baia Mare

MAPPINGS OF PICARD, BESSAGA AND JANOS TYPE

Anton S. MUREŞAN

Abstract. In the paper on consider the mappings of Picard, Bessaga and Janos type. On give the conditions in which the composition of Picard, Bessaga or Janos mappings are of same type.

1. INTRODUCTION

The Picard, Bessaga and Janos mappings are defined from the some theorems which are established in the last thirty years. Thus Bessaga in 1959 [3] prove the following converse of the Banach fixed point theorem:

THEOREM 1.1. Let X be a nonempty set and $f\colon X \longrightarrow X$ a mapping such that $F_{f^k} = \{x^*\}$, for all $k \in \mathbb{N}$. Let $\alpha \in (0,1)$. Then there exists a metric d on X such that:

1°. (X,d) is a complete metric space

2°. $f\colon (X,d) \longrightarrow (X,d)$ is a α-contraction.

Also, Janos in 1967 [7] prove the following converse of the Banach's contraction theorem:

THEOREM 1.2. Let (X,d) be a compact metric space and $f\colon X \longrightarrow X$ a continuous mapping with $\bigcap_{n\in\mathbb{N}} f^n(X) = \{x^*\}$. Then for any $\alpha \in (0,1)$ there exists a metric ρ on X such that:

1°. d and ρ are equivalent

2°. $f: (X,\rho) \longrightarrow (X,\rho)$ is a α-contraction.

In the last years many papers deal with the fixed point theorems, with the converse of Banach fixed point theorem, and with the generalizations or extensions of this (see [1], [2], [5], [6], [10]-[16]). The behavior of some applications having fixed point in respect with the composition or algebraic operations no have approach. In this paper we suggest so subject.

2. DEFINITIONS AND EXAMPLES

DEFINITION 2.1. Let X be a nonempty set. A mapping $f: X \longrightarrow X$ is a **Bessaga mapping** if there exists $x^* \in X$ such that $F_{f^k} = \{x^*\}$ for all $k \in \mathbb{N}$.

EXAMPLE 2.1. Let $X = [0,1]$ and $f: X \longrightarrow X$ defined by

$$f(x) = \begin{cases} 0, & x \in [0,5/8] \\ 2x-5/4, & x \in (5/8,1]. \end{cases}$$

Then f is a Bessaga mapping with $F_{f^k} = \{0\}$, for all $k \in \mathbb{N}$.

DEFINITION 2.2. Let X be a nonempty set. A mapping $f: X \longrightarrow X$ is a **Janos mapping** if $\bigcap\limits_{n\in\mathbb{N}} f^n(X) = \{x^*\}$.

EXAMPLE 2.2. Let $X = \mathbb{R}$ and $f: X \longrightarrow X$ defined by

$$f(x) = \begin{cases} 2x, & x \in \mathbb{R}\setminus[-3,3] \\ x/2, & x \in [-3,3]. \end{cases}$$

Then f is a Janos mapping with $\bigcap\limits_{n\in\mathbb{N}} f^n(X) = \{0\}$.

REMARK 2.1. A Janos mapping is a Bessaga mapping.

DEFINITION 2.3. Let (X,d) be a metric space. A mapping $f: X \longrightarrow X$ is a **Picard mapping** if there exists $x^* \in X$ such that $F_f = \{x^*\}$ and the sequence $(f^n(x_o))_{n\in\mathbb{N}}$ converge to x^* for all $x_o \in X$.

EXAMPLE 2.3. Let $X = [0,1]$ and $f: X \longrightarrow X$ defined by

$$f(x) = \begin{cases} x^2, & x \in [0,1) \\ 1/2, & x = 1. \end{cases}$$

Then f is a Picard mappind with $F_f = \{0\}$.

DEFINITION 2.4. Let (X,d) be a metric space. A mapping $f\colon X \longrightarrow X$ is a **φ-contraction** if there exists a comparison function $\varphi\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ (i.e. monoton increasing and $(\varphi^n(t))_{n\in\mathbb{N}}$ converge to 0 for all $t \geq 0$) such that

$$d(f(x),f(y)) \leq \varphi(d(x,y)) \text{ for all } x,y \in X.$$

EXAMPLE 2.4. Every contraction is a φ-contraction where φ is defined by $\varphi(t) = at$, with $a \in (0,1)$.

REMARK 2.2. A φ-contraction is a Picard mapping.

3. THE MAIN RESULTS

Let (X,d) be a complete metric space. We have the

THEOREM 3.1. Let $\varphi_1,\varphi_2\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

(i) φ_i is monoton increasing for $i = 1,2$

(ii) $\varphi_i(t) < t$ for all $t > 0$ and $i = 1,2$

(iii) φ_i is continuous at the right for $i = 1,2$.

If $f,g\colon (X,d) \longrightarrow (X,d)$ are such that

$$d(f(x),f(y)) \leq \varphi_1(d(x,y))$$

and

$$d(g(x),g(y)) \leq \varphi_2(d(x,y))$$

for all $x,y \in X$, then $f\circ g$ and $g\circ f$ are Picard mappings.

Proof. We have, for all $x,y \in X$,

$$d((f\circ g)(x)(f\circ g)(y)) \leq (\varphi_1\circ\varphi_2)(d(x,y))$$

and for $\varphi_1\circ\varphi_2$ holds the conditions (i), (ii), (iii). Thus, from the Theorem 3.3.3 [13], we get that $f\circ g$ is a Picard mapping. Analogous we prove that $g\circ f$ is a Picard mapping.

REMARK 3.1. If the mappings f and g satisfies the conditions of Theorem 3.1 then $f\circ g$ are Bessaga mappings.

THEOREM 3.2. Let $\varphi_1,\varphi_2\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ comparison functions such that $\varphi_1\circ\varphi_2 = \varphi_2\circ\varphi_1$. If $f\colon (X,d) \longrightarrow (X,d)$ is a φ_1-contraction and $g\colon (X,d) \longrightarrow (X,d)$ is a φ_2-contraction then $f\circ g$ and $g\circ f$ are Picard mappings.

Proof. On can prove easily that $f\circ g$ is a $\varphi_1\circ\varphi_2$-contraction. But $\varphi_1\circ\varphi_2$ is a comparison because $\varphi_1\circ\varphi_2$ is monoton increasing and $((\varphi_1\circ\varphi_2)^n(t))_{n\in\mathbb{N}}$ is a sequence which converge to 0, for all $t \geq 0$. Really, we have $(\varphi_1\circ\varphi_2)^n(t) = (\varphi_1^n\circ\varphi_2^n)(t) = \varphi_1^n(\varphi_2^n(t)) \longrightarrow 0$ as $n \longrightarrow \infty$.

The mapping $g\circ f$ is a $\varphi_1\circ\varphi_2$-contraction too. Thus $f\circ g$ and $g\circ f$ are Picard mappings.

THEOREM 3.3. Let $\psi_1,\psi_2\colon \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\psi_i(t) > 0$ for $t > 0$, when $i = 1,2$. If $f,g\colon (X,d) \longrightarrow (X,d)$ are mappings such that $d(f(x),f(y)) \le d(x,y)-\psi_1(d(x,y))$ and $d(g(x),g(y)) \le d(x,y)-\psi_2(d(x,y))$ for all $x,y \in X$, then $f\circ g$ and $g\circ f$ are Picard mappings.

Proof. We have $d((f\circ g)(x),(f\circ g)(y)) \le d(g(x),g(y))-\psi_1(d(g(x),g(y))) \le d(x,y)-\psi_2(d(x,y))-\psi_1(d(g(x),g(y))) \le d(x,y)-\psi_2(d(x,y))$. Therefore $f\circ g$ satisfie the conditions of Theorem 3.3.5 [13] in respect to the function ψ_2, and such $f\circ g$ is a Picard mapping. The mapping $g\circ f$ satisfie the conditions of Theorem 3.3.5 [13] in respect to the function ψ_1, therefore $g\circ f$ is a Picard mapping.

THEOREM 3.4. Let $\psi_1,\psi_2\colon \mathbb{R}_+ \longrightarrow [0,1)$ monoton decreasing. If $f,g\colon (X,d) \longrightarrow (X,d)$ are such that, for all $x,y \in X$, $d(f(x),f(y)) \le \psi_1(d(x,y))d(x,y)$ and $d(g(x),g(y)) \le \psi_2(d(x,y))d(x,y)$, then $f\circ g$ and $g\circ f$ are Picard mappings.

Proof. We have $d((f\circ g)(x),(f\circ g)(y)) \le \psi_1(d(g(x),g(y)))d(g(x),g(y)) \le \psi_1(d(g(x),g(y)))\psi_2(d(x,y))d(x,y)) \le \psi_2(d(x,y))d(x,y)$. Now the theorem result from the Theorem 3.3.4 [13].

REMARK 3.2. If the mappings f and g satisfies the conditions of Theorems 3.2 or 3.3 or 3.4 then $f\circ g$ and $g\circ f$ are Bessaga mappings.

REMARK 3.3. If the metric space (X,d) is compact and the mappings f and g satisfies the conditions of one from the Theorems 3.1-3.4 then $f\circ g$ and $g\circ f$ are Janos mappings.

For others results on the Picard, Bessaga and Janos mappings see [5], [9], [10], [13].

REFERENCES

1. M. Albu, A fixed point theorem of Maia-Perov type, Studia Univ. Babeş-Bolyai, Ser. Math., 1(1978), 76-79.
2. D.K. BAYEN, Some remarks on fixed point theorems, Mathematica, Tome 29 (52), 1(1987), 1-5.

3. C. Bessaga, On the converse of the Banach fixed point principle, Colloq. Math., 7(1959), 41-43.

4. B.C. Dhage, On extension of a fixed point theorems of Maia, Pure Appl. Math. Sci., XXIV, 1-2(1986), 65-69.

5. B.C. Dhage, On common fixed point mappings in Banach spaces, Jour. Math. Phy. Sci., 22, 3(1988), 431-437.

6. L. Coroian, α_{DP}-condensing mappings and Picard mappings, University of Cluj-Napoca, Research Seminars, Preprint nr. 3, 1988, 17-22.

7. L. Janos, A converse of Banach's contraction theorem, Proc. Amer. Math. Soc., 18(1967), 287-289.

8. M.G. Maia, Un osservazione sulle contrazioni metriche, Rend. Sem. Mat. Univ. Padova, 40(1968), 139-143.

9. A.S. Mureşan, Asupra unor aplicaţii de tip Janos, Vol. Culegere de Studii şi Cercetări Economice, XV, 1985, 251-254.

10. A.S. Mureşan, Some fixed point theorems of Maia type, University of Cluj-Napoca, Research Seminars, Preprint nr. 3, 1988, 35-42.

11. V. Popa, common fixed points of commuting mappings, Mathematica, Tome 29(52), 1(1987), 67-71.

12. V. Popa, Fixed point theorems for commuting mappings, Demonstratio Mathematica, XXI, 1(1988), 143-151.

13. I.A.Rus, Generalized contractions, University of Cluj-Napoca, Research Seminars, Preprint nr. 3, 1983, 1-130.

14. I.A. Rus, A general fixed point principle, University of Cluj-Napoca, Research Seminars, Preprint nr. 3, 1985, 69-76.

15. I.A.Rus, A.S.Mureşan, Examples and counterexamples for Janos mappings, University of Cluj-Napoca, Research Seminars, Preprint nr. 3, 1984, 63-66.

16. B. Rzepecki, Remarks on the Banach fixed point principle and its applications, Rev. Roum. Math. Pures Appl., Tome XXVII,1(1982), 81-93.

Received 02.09.1996

"Babeş-Bolyai" University
Faculty of Mathematics and Informatics
str. Kogălniceanu, 1
RO-3400 Cluj-Napoca
ROMANIA