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POWER SERIES SOLUTIONS OF SOME CAUCHY PROBLEMS

by

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Abstract. In [2], W. Walter introduces a Banach algebra of power series and gives two applications of this algebra, one for an existence and uniqueness proof of a solution for an initial value problem and the other a proof of the implicit function theorem.

The purpose of this note is to show that the method from [2] can be used to prove the existence and uniqueness of power series solutions of some Cauchy problems for first order differential equations and system of differential equations with linear deviating of the argument.

1. THE BANACH ALGEBRA H_r

Let be $r > 0$. We consider the vector space of all functions $u: [-r,r] \to \mathbb{R}$ allowing a power series expansion which is absolutely convergent for $x = r$. We note

$$H_r = \{ u \mid u(x) = \sum_{k=1}^{\infty} C_k x^k \text{, the power series being absolutely convergent for } x = r\}$$

Obviously, the power series is unformly convergent for $|x| \le r$. Let be $\|\cdot\|$ a norm in H_r defined by

$$\|u\| = \sum_{k=0}^{\infty} |C_k| r^k \quad for \quad u \in H_r,\ u(x) = \sum_{k=0}^{\infty} C_k x^k.$$

Then H_r becomes a Banach algebra with a norm of a Banach algebra. That is $\|uv\| \le \|u\| \, \|v\|$ for all $u, v \in H_r$.

The following assertions are true:

(A) $\|x^k\| = r^k$ for $k = 0,1,\ldots$, in particular $\|1\| = 1$;

(B) $u \in H_r$ implies $u^k \in H_r$ and $\|u^k\| \le \|u\|^k$ for $k = 0,1,\ldots$;

(C) If $(u_n)_{n\ge1}$ is a sequence in H_r such that $\sum_{n\ge1}\|u_n\|<\infty$, then $u=\sum_{n\ge1} u_n$ belongs to H_r and $\|u\|\le\sum_{n\ge1}\|u_n\|$.

(D) For the integration operator $I:H_r \to H_r$, defined by $(Iu)(x)=\sum_{k=0}^{\infty} C_k\frac{x^{k+1}}{k+1}$ for $u \in H_r$, $u(x)=\sum_{k=0}^{\infty} C_k x^k$, takes place the inequality $\|Iu\| \le r\|u\|$, with equality for $u = 1$. Hence $\|I\| = r$.

REMARK 1. If $u \in H_r$, $u = u(x)$, and $0 < \lambda < 1$, then $\tilde{u}$ where $\tilde{u}(x) = u(\lambda x)$ belongs to H_r and we have

$$\|\tilde{u}\| \le \|u\|$$

2. A CAUCHY PROBLEM FOR A FIRST ORDER DIFFERENTIAL EQUATION WITH LINEAR DEVIATING OF THE ARGUMENT

Consider the problem

(1) $y'(x) = f(x,y(\lambda x))$, $x \in [-r,r]$, $r > 0$, $0 < \lambda < 1$

(2) $y(0) = 0$,

where $f:D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, $f = f(x,t)$ and

$$f(x,t)=\sum_{i,j\ge0} a_{ij}\, x^i t^j, \text{ with } a_{ij} \in \mathbb{R}.$$

We search solutions in H_r of the problem (1) + (2).

The problem (1)+(2) is equivalent with the following integral equation

(3) $y(x) = \int_0^x f(s,y(\lambda s))ds$, $x \in [-r,r]$

Let be $b > 0$. Consider the ball

$\bar{B}(0;b) = \{u \in H_r \mid \|u\| \le b\}$.

We have

THEOREM 1 We suppose that:

(i) $f(x,t)=\sum_{i,j\ge0} a_{ij}\, x^i t^j$ where $a_{ij} \in \mathbb{R}$;

(ii) $r > 0$ and $b > 0$ are so that we have

$$M = \sum_{i,j\ge0} |a_{ij}| r^i b^j < \infty$$

and

$$L = \sum_{i,j\ge0} |a_{ij}| r^i j b^{j-1} < \infty;$$

(iii) $r \leq b/M$ and $r < 1/L$.

Then the problem (1) + (2) has in $\bar{B}(0;b)$ an unique solution.

Proof We consider the operator $A:\bar{B}(0;b) \to H_r$ defined by

$(Ay)(x) = \int_0^x f(s,y(\lambda s))ds \ , \quad x \in [-r,r]$.

Each solution of the integral equation (3) is a fixed point for the operator A.

We show that the operator A maps the closed ball $\bar{B}(0;b)$ into itself and that it is a contraction.

Let be $y \in \bar{B}(0;b)$. Then $\tilde{y} \in \bar{B}(0;b)$ where $\tilde{y}(x) = y(\lambda x)$. We note $(Fy)(s) = f(s, \tilde{y}(s))$. The function v_k defined by

$$v_k(s) = \sum_{i+j=k} a_{ij}\, s^i (\tilde{y}(s))^j$$

belongs to H_r and from (A) and (B) the estimate

$$\|v_k\| \leq \sum_{i+j=k} |a_{ij}| r^i b^j$$

follows. Since $\sum_{k\geq 0} \|v_k\| \leq M$,

if follows from (C) that $\sum_{k\geq 0} v_k = Fy$ belongs to H_r and $\|Fy\| \leq M$.

Hence by (D) and (iii) we have that

$\|Ay\| = \|I\ Fy\| \leq r\ M \leq b$.

This shows that $A(\bar{B}(0;b)) \subseteq \bar{B}(0;b)$ that is

$A:\bar{B}(0;b) \to \bar{B}(0;b)$.

Now, let be $y_1, y_2 \in \bar{B}(0;b)$. The decomposition

$$y_1^j - y_2^j = (y_1 - y_2)(y_1^{j-1} + y_1^{j-2} y_2 + \ldots + y_2^{j-1})$$

implies that

$\|y_1^j - y_2^j\| \leq \|y_1 - y_2\|\ j\ b^{j-1}$, for $j = 1,2,\ldots$.

and hence

$$\|Fy_1 - Fy_2\| \leq \sum_{i,j\geq 0} |a_{ij}| \|s^i\| \|y_1^j - y_2^j\| \leq$$

$$\leq \sum_{i,j\geq 0} |a_{ij}| r^i j b^{j-1} \|y_1 - y_2\| = L\|y_1 - y_2\|.$$

According to (D) we have

$\|Ay_1 - Ay_2\| \leq rL\|y_1 - y_2\|$ for every $y_1, y_2 \in \bar{B}(0;b)$.

By (iii), $rL < 1$, so that the operator A is a contraction.

Now, the theorem follows from the Contraction principle.

REMARK 2 The constant M is an upper bound for $|f|$, $f = f(x,t)$, in the region $|x| \le r$, $|t| \le b$, and the restriction $r \le b/M$ insures the invariance of the ball.
The consider $r < 1/L$ insures the A is a contraction. Here L is an upper bound for $\left|\frac{\partial f}{\partial t}\right|$ in the same region $|x| \le r$, $|t| \le b$. If the condition $r < 1/L$ it not satisfied, we choose r_1 such that $0 < r_1 < 1/L \le r$. As in REMARK 3 of [2] one can get rid of the condition $r < 1/L$.

3. A CAUCHY PROBLEM FOR A FIRST ORDER SYSTEM OF DIFFERENTIAL EQUATIONS WITH LINEAR DEVIATING OF THE ARGUMENT

Consider the problem

(4) $y'(x) = f(x,y(\lambda x))$, $x \in [-r,r]$, $0 < \lambda < 1$

(5) $y(0) = (0,0,\ldots,0)$,

where $f = f(x,t)$, $t = (t_1,t_2,\ldots,t_n)$, $f = (f_1,f_2,\ldots,f_n)$ and

$$f(x,t) = \sum_{\substack{j\ge 0\\ p\ge 0}} a_{jp}\, x^j t^p, \quad with \quad a_{jp} \in \mathbb{R}^n.$$

We introduce the Banach space H_r^n of functions $w=(w_1,w_2,\ldots,w_n)$, where $w_i \in H_r$ for every $i=\overline{1,n}$ and use the maximum norm

$$\|w\|_n = \max_{i=\overline{1,n}} \|w_i\|$$

The following assertions are true:

(E) $u \in H_r$, $w \in H_r^n$ implies $uw = (uw_i)_{i=\overline{1,n}} \in H_r^n$ and

$$\|uw\|_n \le \|u\| \cdot \|w\|_n \ ;$$

(F) For $w \in H_r^n$ we have that, $w^p = w_1^{p_1}\ldots w_n^{p_n} \in H_r$, where $p = (p_1,\ldots,p_n)$ is a multi-index with $|p| = p_1 + \ldots + p_n$, $p_i \ge 0$, and w^p satisfies the following inequality:

$$\|w^p\| \le \|w\|_n^{|p|} \ ;$$

(G) If $w_1, w_2 \in H_r^n$ and $\|w_1\|_n \le b_1$, $\|w_2\|_n \le b_1$ then we have

$$\|w_1^p - w_2^p\| \le \|w_1 - w_2\|_n \ |p| \ b_1^{|p|-1}$$

REMARK 3 If $w = (w_1,\ldots,w_n) \in H_r^n$ and $0 < \lambda < 1$ then $\tilde{w} = (\tilde{w}_1,\ldots,\tilde{w}_2) \in H_r^n$, where $\tilde{w}_i(x) = w_i(\lambda x)$, for every $x \in [-r,r]$ and $i=\overline{1,n}$.

THEOREM 2 We suppose that:

(i) $f(x,t)=\sum_{\substack{j\geq 0\\ p\geq 0}} a_{jp}\, x^j t^p$, with $a_{jp} \in \mathbb{R}^n$, where

$f = (f_1, f_2, \ldots, f_n)$ and $t = (t_1, \ldots, t_n)$.

(ii) there exists $b > 0$ such that

$$M = \sum_{\substack{j\geq 0,\, p=(p_1,\ldots,p_n)\\ |p|=p_1+\ldots+p_n}} |a_{jp}| r^j b^{|p|} < \infty,$$

where $|\cdot|$ denotes the maximum norm in $\mathbb{R}^n$.

Then the problem (4) + (5) has an unique power series solution which converges (at least) for every x so that $|x| < e$ where $e = \min(r, b/M)$.

<u>Proof</u> The proof runs as in Theorem 3, [2]. We use REMARK 3. Let $0 < c < b$ and $r_2 = \min(r, c/M)$. Then

$$L = \sum_{\substack{j\geq 0\\ |p|\geq 0}} |a_{jp}| r_2^j |p| c^{|p|-1} < \infty,$$

since $|p| c^{|p|-1} < b^{|p|}$ for large $|p|$.

We consider the ball

$$\bar{B}(0;c) = \{w \in H^n_{r_2} \mid \|w\|_n \leq c\}$$

and the operator $A:\bar{B}(0;c) \to H^n_{r_2}$ defined by

$$(Ay)(x) = \int_0^x f(s, y(\lambda s))ds, \quad x \in [-r_2, r_2].$$

Using (E) - (G) it is shown as in Theorem 1 that the operator A maps $\bar{B}(0;c)$ into itself.

For $w_1, w_2 \in \bar{B}(0;c)$ the following Lipschitz condition holds

$$\|Aw_1 - Aw_2\|_n \leq L\, \|w_1 - w_2\|_n.$$

If r_2 satisfies also $r_2 L < 1$, then A is a contraction.

The Contraction principle guarantees that an unique solution $y^* \in H^n_{r_2}$ of the problem (4)+(5) exists. Since c can be arbitrarily close to b, the theorem follows.

REFERENCES

1. I.A. Rus, Principii şi aplicaţii ale teoriei punctului fix, Editura Dacia, Cluj-Napoca, 1979
2. W. Walter, A useful Banach algebra, Elemente der Mathematik, 47(1992), 27-32

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