Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol. XII (1996), 163-168

ON THE PRIME RADICAL OF AN IDEAL IN AN (m,n)-RING

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Abstract. The extension of the usual ring concept to the case where the underlying group and semigroup are respectively an m-ary group and an n-ary semigroup has been studied by Crombez [1]; some ideal theory aspects and the properties of the prime radical of an ideal in a commutative (m,n)-ring for $m=n$ were investigated. In this note we prove that these properties remain true for $n\neq m$, too.

1. Definitions, notations and preliminary results

An algebra $(R, +, \circ)$ is an *(m,n)-ring*, $m,n\in\mathbb{N}^*\setminus\{1\}$ if:

1) $(R,+)$ is a commutative m-group;

2) $(R,\circ)$ is an n-semigroup;

3) the following distributive laws hold for all choices of $a_1,\ldots,a_n$, $b_1,\ldots,b_m\in R$ and for all choices of $i=\{1,2,\ldots,n\}$:

$$(1)\quad \begin{aligned}&(a_1,\ldots,a_{i-1},(b_1+\ldots+b_m),a_{i+1},\ldots,a_n)_\circ=\\&=(a_1,\ldots,a_{i-1},b_1,a_{i+1},\ldots,a_n)_\circ+\ldots+(a_1,\ldots,a_{i-1},b_m,a_{i+1},\ldots,a_n)_\circ\end{aligned}$$

If, in particular, the n-ary operation (multiplication) is commutative, we call $(R,+,\circ)$ a commutative (m,n)-ring.

In keeping the practice adopted for polyadic semigroups briefly notational convenience will be used, as follows:

$$x_1+\ldots+x_j+\underbrace{x+\ldots+x}_{k\text{-times}}+x_{j+k+1}+\ldots+x_m=\sum_{i=1}^{j}x_i+kx+\sum_{i=j+k+1}^{m}x_i$$

and

$$(x_1,\ldots,x_j,\underbrace{x,\ldots,x}_{k\text{-times}},x_{j+k+1},\ldots,x_m)_o=(x_1^j,\overset{(k)}{x},x_{j+k+1}^n)_o$$

Clearly that $\sum_{i=k}^{j}x_i$ and x_k^j, for $k>j$ denote empty sequences.

Also, in writing long products (i.e. products having $p\equiv 1 \pmod{n-1}$ factors) and long sums ($p\equiv 1 \pmod{m-1}$ terms) we shall omit supplementary brackets (sum symbols $\sum$).

With these conveniences the distributive laws can be written

$$\text{(1')}\qquad \left(a_1^{i-1},\sum_{j=1}^{m}b_j,a_{i+1}^{m}\right)_o=\sum_{j=1}^{m}\left(a_1^{i-1},b_j,a_{i+1}^{n}\right)_o$$

We may also denote recursively (Dörnte [2]):

$$a^{[o]}=a;\ a^{[1]}=ma;\ a^{[k]}=(m-1)a+a^{[k-1]}=(km-k+1)a$$

and

$$a^{\langle o\rangle}=a;\ a^{\langle 1\rangle}=\left(\overset{(n)}{a}\right)_o;\ a^{\langle k\rangle}=\left(a^{\langle k-1\rangle},\overset{(n-1)}{a}\right)_o=\left(\overset{(kn-k+1)}{a}\right)_o,\text{for } k\in\mathbb{N}^*$$

It is easily verified that

$$\left(a^{\langle k_1\rangle},a^{\langle k_2\rangle},\ldots,a^{\langle k_n\rangle}\right)_o=a^{\langle k_1+\ldots+k_n+1\rangle},$$

and

$$\left(a^{\langle k\rangle}\right)^{\langle r\rangle}=a^{\langle kr(n-1)+k+r\rangle}=\left(a^{\langle r\rangle}\right)^{\langle k\rangle}$$

for every choice of the natural numbers $k_1,\ldots,k_n,k,r$.

Also we may conclude that

Proposition 1. In a commutative (m,n)-*ring* $(R,+,o)$ the following exponential laws are verified

$$\text{(4)}\qquad \left(\sum_{i=1}^{m}a_i\right)^{\langle k\rangle}=\sum_{\substack{k_1+\ldots+k_m=k(n-1)+1\\ 0\le k_i\le k(n-1)+1\\ i=1,2,\ldots,m}}\frac{[k(n-1)+1]!}{k_1!\ldots k_m!}\left(\overset{(k_1)}{a_1}\ldots\overset{(k_m)}{a_m}\right)_o$$

for all $a_1,\ldots,a_m \in R$ *and* $k\in\mathbb{N}$.

An element $a\in R$ is called an **additive (multiplicative) idempotent** if $a^{[1]}=a$ $(a^{\langle 1\rangle}=a)$ and **idempotent** if both of these conditions are satisfied. An element $z\in R$ is called a **zero** of R if $(x_1^{i-1},z,x_{i+1}^n)_o=z$ for all $x_1,\ldots,x_n\in R$ and for all choices of $i\in\{1,2,\ldots,n\}$.
A zero, if there exists is clearly an idempotent of R; an (m,n)-*ring* may have at most one zero. If R is a $(2,n)$-*ring*, then R has a zero element.

The element $\overline{a}$ will denote the aditive querelement of $a\in R$, *i.e.* $\overline{a}$ is the solution of the equation $(m-1)a+x=a$.
It is easily seen that in an (m,n)-ring we have

$$\overline{\sum_{i=1}^{m} a_i}=\sum_{i=1}^{m}\overline{a_i},\quad \forall a_1,\ldots,a_m\in R \tag{2}$$

$$\overline{(a_1^n)_o}=(a_1^{i-1},\overline{a_i},a_{i+1}^n)_o,\quad \forall a_1,\ldots,a_n\in R;\ \forall i\in\{1,\ldots,n\}. \tag{3}$$

Crombez, in [1], defined an i-ideal of an (m,n)-ring R as a non empty subset $U\subseteq R$, such that $(U,+)$ is sub-m-group of $(R,+)$ and $\left(\overset{(i+1)}{R}\ U\ \overset{(n-i)}{R}\right)_o\subseteq U$, where i has one of the values $1,2,\ldots,m$. If U is an i-ideal for each $i=1,2,\ldots,n$, then it is simply called **ideal** of R.
An ideal U *of* R is called **completely prime** iff $(x_1^n)_o\in U$ imply $x_i\in U$ for some $i\in\{1,2,\ldots,n\}$, and **primary** if $(x_1^n)_o\in U$ and $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n\notin U$, imply the existence of a natural number p such that $x_i^{\langle p\rangle}\in U$. Hence, each completely prime ideal is a primary ideal. An ideal P is called **prime** iff for any

ideals $U_1,\ldots,U_m$ with $(U_1^m)_o \subseteq P \Longrightarrow U_i \subseteq P$ for some $i\in\{1,\ldots,m\}$. Each completely prime ideal is a prime ideal.

2. The prime radical of an ideal

Definition [1] 1) Let $(R,+,\circ)$ be an (m,n)-ring and U an ideal of R. **The prime radical** $\sqrt{U}$ is the set

$$\sqrt{U}=\{x\in R \mid x^{\langle p\rangle}\in U \ \ for \ \ some \ \ p\in\mathbb{N}\}.$$

2) An ideal U of R is called a **radical ideal** if $U=\sqrt{U}$.

Remarks. 1) $U\subseteq\sqrt{U}$.

2) The intersection of an arbitrary family of completely prime ideals of an (m,n)-ring is a radical ideal.

Proposition 2. *The prime radical $\sqrt{U}$ of an ideal U of a commutative (m,n)-ring R is an ideal of R.*

Proof. Let $a_1,\ldots,a_m\in\sqrt{U}$, *i.e.* $\exists p_1,\ldots,p_m\in\mathbb{N}$ such that $a_i^{\langle p_i\rangle}\in U$; $i=1,\ldots,m$. There exists $k\in\mathbb{N}$, $k=p_1+\ldots+p_m+\alpha$, where $\alpha=\left[\dfrac{m-1}{n-1}\right]$ if $n-1$ is a divisor of $m-1$ and $\alpha=1+\left[\dfrac{m-1}{n-1}\right]$ otherwise, so that $\left(\sum\limits_{i=1}^{m} a_i\right)^{\langle k\rangle}\in U$, hence $\sum\limits_{i=1}^{m} a_i\in U$. Indeed, by proposition 1, the equality (4) holds.

In every long product $\left(\overset{(k_1)}{a_1},\ldots,\overset{(k_m)}{a_m}\right)_o$ there is an $i_o\in\{1,2,\ldots,m\}$ so that $k_{i_o}\geq p_{i_o}(n-1)+1$. Otherwise, if $k_i<p_i(n-1)+1$ for all $i=1,\ldots,m$, then $\sum\limits_{i=1}^{m} k_i<\left(\sum\limits_{i=1}^{m} p_i\right)(n-1)+m$, *i.e.*

$k(n-1)+1<\left(\sum_{i=1}^{m} p_i\right)(n-1)+m$, hence $\alpha(n-1)<m-1$; contradiction to the choice of $\alpha\in\mathbb{N}$.

If $r_{i_o}=k_i-p_{i_o}(n-1)-1$, because U is an ideal of R we have

$$\left(\overset{(k_1)}{a_1},\ldots,\overset{(k_{i_o})}{a_{i_o}},\ldots,\overset{(k_m)}{a_m}\right)_o=\left(\overset{(k_1)}{a_1},\ldots,\overset{(r_{i_o})}{a_{i_o}},a_{i_o}^{\langle p_{i_o}\rangle},\ldots,\overset{(k_m)}{a_m}\right)_o\in U$$

and $\left(\sum a_i\right)^{\langle k\rangle}\in U$.

If $a\in\sqrt{U}$, than $a^{\langle p\rangle}\in U$ for some $p\in\mathbb{N}$. Since $(U,+)$ is a sub-m-group of $(R,+)$, we have $\overline{a^{\langle p\rangle}}\in U$ and $\overline{\overline{a^{\langle p\rangle}}}\in U$. But, by (3) we find recursively for $k=p(n-1)+1$ that

$$\overline{\overline{a^{\langle p\rangle}}}=\overline{a^{\langle p\rangle}},$$

hence $\overline{a}\in\sqrt{U}$.

Also, for all $x_1,\ldots,x_{n-1}\in R$ and $a\in\sqrt{U}$, since we suppose the n-ary operation to be commutative, we have $\left(x_1^{n-1},a\right)_o^{\langle p\rangle}=\left(x_1^{\langle p\rangle},\ldots,x_{n-1}^{\langle p\rangle},a^{\langle p\rangle}\right)_o\in U$, hence $\left(x_1^{n-1},a\right)_o\in\sqrt{U}$.

Then it follows that $\sqrt{U}$ is an ideal of R.
In the same manner as in [1] we may show that **in a commutative (m,n)-ring** $(R,+,\circ)$ holds the following

Proposition 3. *If U and V are ideals of R and $U^{\langle k\rangle}\subseteq V$ for some natural number k, then $\sqrt{U}\subseteq\sqrt{V}$.*

Proposition 4. *If $U_1,\ldots,U_n$ are ideals of R, then*

$$\sqrt{(U_1,\ldots,U_n)_o}=\sqrt{\bigcap_{i=1}^{n} U_i}=\bigcap_{i=1}^{n}\sqrt{U_i}$$

The following proposition can be proved in a similar way as proposition 2.

Proposition 5. *If $U_1,\ldots,U_m$ are ideals of R, then*

$$\sqrt{\sum_{i=1}^{m} U_i}=\sqrt{\sum_{i=1}^{m} \sqrt{U_i}}.$$

Proposition 6. *The radical of a primary ideal U of R is a completely prime ideal contained in each completely prime ideal containing U.*

Proposition 7. *If U and V are ideals in a commutative (m,n)-ring R, then U is primary and $V=\sqrt{U}$ if and only if the following conditions are satisfied:*

1º $U\subseteq V$

2º $x\in V \rightarrow \exists p\in\mathbb{N};\ x^{(p)}\in U$

3º $(a_1^n)_o \in U$ and $a_1,\ldots,a_{n-1}\notin V$ imply $a_n\in U$.

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Received at: 02.09.1996

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