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On some operators of blending type

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0. Beginning with the paper by Barnhill, Birkhoff and Gordon[1], the interpolation problem to boundary data on a triangle was largely studied. Important contributions to the development of this theory are due to Gh. Coman, I.Gânscă and L.Ţâmbulea [4], [5],[6]. So, in the paper [6] there are constructed some interpolants for a given function on the sides of a triangle T and on one of its median.

In this paper we will present a slight generalization: we will construct some interpolants for a given function on the sides of a triangle T and on one of his cevian. In particular this cevian can be a median, a bisectrix etc.

1. Let T be the standard triangle $T_a=\{(x,y)\in\mathbb{R}^2 \mid x\geq 0, y\geq 0, x+y\leq a\}$

with the vertex $V_1=(a,0)$, $V_2=(0,a)$, $V_3=(0,0)$ and with the side oposite to V_k denoted by E_k, $k=\overline{1,3}$ (fig.1).

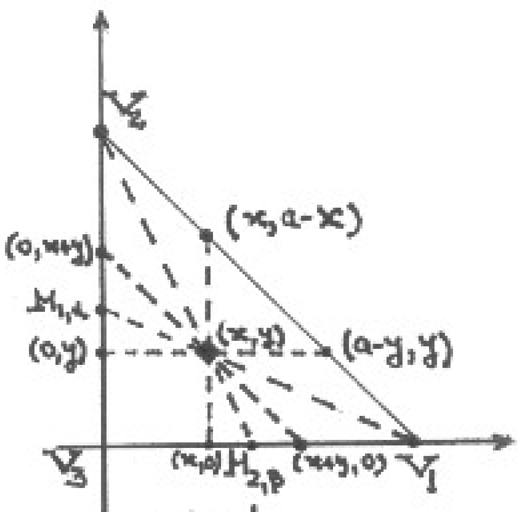

fig.1

Let $f:T_a\to\mathbb{R}$ be a given function and let L_1^x, L_1^y be the linear interpolation operators along the parallels to the sides E_1 respectively E_2 of T_a, i.e:

$$(L_1^x f)(x,y)=\frac{a-x-y}{a-x}f(x,0)+\frac{y}{a-x}f(x,a-x)$$

$$(L_1^y f)(x,y)=\frac{a-x-y}{a-y}f(0,y)+\frac{x}{a-y}f(a-y,y)$$

It is well known that each of these operators interpolates the function f along the two sides of T_h:

$$(L_1^y f)(x,0)=f(x,0),\ (L_1^y f)(x,a-x)=f(x,a-x),\ x\in[0,a]$$
$$(L_1^x f)(0,y)=f(0,y),\ (L_1^x f)(a-y,y)=f(a-y,y),\ y\in[0,a].$$

Now, we suppose that $\alpha,\beta\in(0,1)$ and we denote by $L_{2,\alpha}^y$ and $L_{2,\beta}^x$ the quadratic interpolation operators along the parallels to E_1 respectively E_2, that interpolate the function f in $(x,0)$, $(x,\alpha(a-x)),(x,a-x)$ respectively $(0,y),(\beta(a-y),y),(a-y,y)$.

The point $(x,\alpha(a-x))$ lie to the cevian $V_1M_{1,\alpha}$ and the point $(\beta(a-y),y)$ lie to the cevian $V_2M_{2,\beta}$.

We have

$$(L_{2,\alpha}^y f)(x,y)=\frac{(a-x-y)(\alpha a-\alpha x-y)}{\alpha(a-x)^2}f(x,0)+\frac{y(a-x-y)}{\alpha(1-\alpha)(a-x)^2}f(x,\alpha a-\alpha x)+$$
$$+\frac{y(y+\alpha x-\alpha a)}{(1-\alpha)(a-x)^2}f(x,a-x).$$

and

$$(L_{2,\beta}^x f)(x,y)=\frac{(a-x-y)(\beta a-\beta y-x)}{\beta(a-y)^2}f(0,y)+\frac{x(a-x-y)}{\beta(1-\beta)(a-y)^2}f(\beta a-\beta y,y)+$$
$$+\frac{x(x+\beta y-\beta a)}{(1-\beta)(a-y)^2}f(a-y,y)$$

So

$$(L_{2,\alpha}^y f)(x,0)=f(x,0),\ (L_{2,\alpha}^y f)(x,a-x)=f(x,a-x),\ (L_{2,\alpha}^y f)(x,\alpha(a-x))=f(x,\alpha(a-x))$$

with $x\in[0,a]$, i.e. $L^y_{2,\alpha}$ interpolates the function f on the sides E_1 and E_3 of T_h and on the cevian $V_1M_{1,\alpha}$. In a similar way $L^x_{2,\beta}$ interpolates the function f on the sides E_1 and E_2 and on the cevian $V_2M_{2,\beta}$.

Theorem 1.1. If $f\in C(T_a)$ then

$$L^y_{2,\alpha}\oplus L^x_1 f=f \text{ on } \partial T_a\cup V_1M_{1,\alpha}$$
$$L^x_{2,\beta}\oplus L^y_1 f=f \text{ on } \partial T_a\cup V_2M_{2,\beta}$$

Proof:

Taking account that

$$(L^y_{2,\alpha}\oplus L^x_1 f)(x,y)=L^y_{2,\alpha}f(x,y)+L^x_1 f(x,y)-L^y_{2,\alpha}L^x_1 f(x,y)$$

and

$$(L^y_{2,\alpha}L^x_1 f)(x,y)=\frac{(a-x-y)(\alpha a-\alpha x-y)}{\alpha(a-x)^2}\cdot\left(\frac{a-x}{a}f(0,0)+\frac{x}{a}f(a,0)\right)+$$
$$+\frac{y(a-x-y)}{\alpha(1-\alpha)(a-x)^2}\cdot\left(\frac{(1-\alpha)(a-x)}{(1-\alpha)a+\alpha x}f(0,\alpha a-\alpha x)+\frac{x}{(1-\alpha)a+\alpha x}f((1-\alpha)a+\alpha x,\alpha a-\alpha x)\right)+$$
$$+\frac{y(\alpha x+y-\alpha a)}{(1-\alpha)(a-x)^2}f(x,a-x)$$

we obtain:

$$(L^y_{2,\alpha}\oplus L^x_1 f)(x,y)=\frac{a-x-y}{(a-x)^2}\left[\frac{\alpha a-\alpha x-y}{\alpha}f(x,0)+\frac{y}{\alpha(1-\alpha)}f(x,\alpha a-\alpha x)\right]+$$
$$+\frac{1}{a-y}[(a-x-y)f(0,y)+xf(a-y,y)]-\frac{(a-x-y)(\alpha a-\alpha x-y)}{\alpha a(a-x)}\left[f(0,0)+\frac{x}{a-x}f(a,0)\right]-$$
$$-\frac{y(a-x-y)}{\alpha(1-\alpha)(a-x)[(1-\alpha)a+\alpha x]}\left[(1-\alpha)f(0,\alpha a-\alpha x)+\frac{x}{a-x}f((1-\alpha)a+\alpha x,\alpha a-\alpha x)\right].$$

In a similar way:

$$(L^x_{2,\beta}\oplus L^y_1 f)(x,y)=\frac{a-x-y}{(a-y)^2}\left[\frac{\beta a-x-\beta y}{\beta}f(0,y)+\frac{x}{\beta(1-\beta)}f(\beta a-\beta y,y)\right]+$$
$$+\frac{1}{a-x}[(a-x-y)f(x,0)+yf(x,a-x)]-\frac{(a-x-y)(\beta a-x-\beta y)}{\beta a(a-y)}\left[f(0,0)+\frac{y}{a-y}f(0,a)\right]-$$
$$-\frac{x(a-x-y)}{\beta(1-\beta)(a-y)[(1-\beta)+\beta y]}\left[(1-\beta)f(\beta a-\beta y,0)+\frac{y}{a-y}f(\beta a-\beta y,(1-\beta)a+\beta y)\right]$$

The proof follows by direct substitution.

Theorem 1.2. $L^y_{2,\alpha}\oplus L^x_1 f=f$ and $L^x_{2,\alpha}\oplus L^y_1 f=f$ for any $f\in P^2_2$ (the set of all polynomials with the degree less or equal to 2).

Proof:

As $L_{2,\alpha}\oplus L^x_1$ and $L^x_{2,\alpha}\oplus L^y_1$ are linear operators the two equalities must be verified for the test functions e_{ij}, $i+j\le 2$, where $e_{ij}(x,y)=x^i y^j$. This way the proof is a straight forward computation.

We consider now the following approximation formula:

$$f=L^y_{2,\alpha}\oplus L^x_1 f+R^{yx}_{21,\alpha}f$$

where R^{yx}_{21} is the corresponding remainder term.

Theorem 1.3. If $f\in B_{12}(0,0)$ [7,p.175], then

$$(R^{yx}_{21}f)(x,y)=\int_0^a K_{30}(x,y,s)f^{(3,0)}(s,0)ds+\int_0^a K_{21}(x,y,s)f^{(2,1)}(s,0)ds+$$
$$+\int_0^a K_{03}(x,y,s)f^{(0,3)}(0,t)dt+\iint_{T_a} K_{12}(x,y,s,t)f^{(1,2)}(s,t)dsdt$$

where

$$K_{30}(x,y,s)=\frac{y(y+\alpha x-\alpha a)}{2(1-\alpha)(a-x)^2}(x-s)^2_+-\frac{x}{2(a-y)}(a-y-s)^2_++$$
$$+\frac{x(a-x-y)(\alpha a-\alpha x-y)}{2\alpha a(a-x)^2}(a-s)^2_++\frac{xy(a-x-y)}{2\alpha(1-\alpha)(a-x)^2[(1-\alpha)a+\alpha x]}[(1-\alpha)a+\alpha x-s]^2_+$$
$$K_{21}(x,y,s)=\frac{y(\alpha x+y-\alpha a)}{(1-\alpha)(a-x)}(x-s)_+-\frac{xy}{a-y}(a-y-s)_++\frac{\alpha xy}{(a-x)[(1-\alpha)a+\alpha x]}[(1-\alpha)a+\alpha x-s]_+$$
$$K_{03}(x,y,t)=0$$
$$K_{12}(x,y,s,t)=(x-s)^0_+(y-t)^0_+-\frac{y(a-x-y)}{\alpha(1-\alpha)(a-x)^2}(x-s)^0_+(\alpha a-\alpha x-t)_+-$$
$$-\frac{x}{a-y}(a-y-s)^0_+(y-t)_++\frac{xy}{\alpha(1-\alpha)[(1-\alpha)a+\alpha x](a-x)^2}[(1-\alpha)a+\alpha x-s]^0_+(\alpha a-\alpha x-t)_+$$

Proof:

We have that $R^{yx}_{21,\alpha}f=0\ \forall\ f\in P^2_2$ and the proof follows by the Sard kernels theorem in triangles [7].

The expressions of the kernels are:

$$K_{20}(x,y,s)=R^{x}_{21,\alpha}\left(\frac{(x-s)^2_+}{2}\right)=\frac{(x-s)^2_+}{2}-\left(L^y_{2,\alpha}\oplus L^x_1\frac{(x-s)^2_+}{2}\right)(x,y)$$
$$K_{11}(x,y,s)=R^{x,y}_{21,\alpha}((x-s)_+y)=(x-s)_+y-(L^y_{2,\alpha}\oplus L^x_1(x-s)_+y)(x,y)$$
$$K_{02}(x,y,t)=R^{y}_{21,\alpha}\left(\frac{(y-t)^2_+}{2}\right)=\frac{(y-t)^2_+}{2}-\left(L^y_{2,\alpha}\oplus L^x_1\frac{(y-t)^2_+}{2}\right)(x,y)$$
$$K_{12}(x,y,s,t)=R^{x,y}_{21,\alpha}((x-s)^0_+(y-t)_+)=$$
$$=(x-s)^0_+(y-t)_+-(L^y_{2,\alpha}\oplus L^x_1(x-s)^0_+(y-t)_+)(x,y)$$

Remark 1.1. With the approximation formula $f=L^x_{2,\beta}\oplus L^y_1f+R^{xy}_{21}f$ an analogous theorem can be given for the remainder $R^{xy}_{21}f$.

Remark 1.2. If $\alpha=\beta=\frac{1}{2}$ then the cevians $V_1M_{1,\alpha}$ and $V_2M_{2,\beta}$ are medians and we find the results in paper [6].

Remark 1.3. If $\alpha=\beta=\frac{\sqrt{2}}{2}$ then the cevians $V_1M_{1,\alpha}$ and $V_2M_{2,\beta}$ are bisectrix. In this case we obtain interpolants for a given function on the sides of a triangle T_h and on one of its bisectrix.

Abstract. Some interpolants for a given function on the sides of a standard triangle $T_a=\{(x,y)\in\mathbb{R}|\ x\geq0, y\geq0, x+y\leq a\}$ and on one of its cevian are constructed. The properties of these interpolants are given in the theorems 1.1 and 1.2. The expression of the corresponding remainder term is given in the theorem 1.3. In particular, we find the results from [6] and also interpolants for a given function on the sides of the standard triangle T_a and on one of its bisectrix.

References

1. R.E.Barnhill, G.Birkhoff and W.J.Gordon, Smooth interpolation in triangles, J.Approx.Theory 8(1973), 114-128

2. R.E.Barnhill, L.Mansfield, Error bounds for smooth interpolation in triangles, J.Approx.Theory (1974), 306-318

3. K.Bohmer, Gh.Coman, On some approximation schemes in triangles, Mathematica 22(45)(1980), 231-235.

4. Gh.Coman, Analiză numerică, Ed. Libris, Cluj, 1995

5. Gh.Coman, I.Gânscă and L.Ţâmbulea, Some practical applications of blending interpolation III, Research Seminars, Preprint nr.6, 1989, 5-22

6. Gh.Coman, I.Gânscă and L.Ţâmbulea, New interpolation procedure in triangle, Studia Univ. Babeş-Bolyai, Mathematica, XXXVII, I,1992, 37-45.

7. A.Sard, Linear Approximation, AMS, Providence R.I., 1963.

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