Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol.XII (1996),175-180

Dedicated to the 35th anniversary of the University of Baia Mare

A FEEDBACK SOLUTION OF A LINEAR QUADRATIC PROBLEM FOR BOUNDARY CONTROL OF LAPLACE EQUATION

Maria BĂTINEŢU - GIURGIU

A boundary control problem with quadratic cost functional for *Laplace* equation with boundary condition given by the solution of a differential equation involving control is considered. The form of the feedback for the optimal control is derived. The existence and uniqueness of the solution (in the classical sense) for the *Hamilton* type system is discussed.

1. PRELIMINARIES AND NOTATIONS

Let $\mathcal{U}=L^2([0,2\pi];\Re^k), k\geq 1$ be the space of the control functions. We set $D=\{(\rho,\theta)|\ \rho\in[0,2\pi]\}$. $A(\theta)$, $B(\theta)$ are matrices of continuous functions of $m\times m$ type, respectively of $m\times k$ type; C is a constant vector of $1\times m$ type.

Consider the following boundary problem, defined on D :

(1) $$\Delta y(r,\theta)=0,\ (\Delta=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2})$$

along with boundary condition :

(2) $$y(R,\theta)=f(\theta)$$

(3) $$f(\theta)=Cz(\theta),(\forall)\theta\in[0,2\pi]$$

(4) $$\frac{dz}{d\theta}=A(\theta)z+B(\theta)u(\theta),\ \text{for } u\in\mathcal{U}$$

(5) $$z(\theta)=z_0,(z_0\in\Re^m,\ \text{is given}).$$

For every $u \in \mathscr{U}$, the problem (4) - (5) has a unique solution, let us denote it by $z_u(\theta)$. The solution $z_u(.)$ is an absolute continuous function. According to (3) the boundary data $f(.)$ in the problem (1) - (2) is as smooth as the solution $z_u(.)$. It follows that the problem (1) - (2) has a unique solution; let us denote it by $y_u(r,\theta)$. The function $y_u(.,.)$ is the classical solution of Laplace equation related to the domain D and it depends continuously on the boundary data f.

For every $u \in \mathscr{U}$ fixed, the solution of the problem (1) - (5) consists of the couple $(y_u(r,\theta), z_u(\theta))$.

Let $\Phi(\theta,s)$ be the fundamental matrix of the solutions of the problem (4) - (5) and $G(\rho,\varphi;r,\theta)$ the *Green*'s function corresponding to the *Laplace* operator and the domain D. Then, the solution of the problem (1) - (5) has the representation:

$$(6)\qquad \begin{cases} z_u(\theta) = \Phi(\theta,0)z_0 + \int_0^\theta \Phi(\theta,s)B(s)u(s)\,\mathrm{d}s \\ f(\theta) = Cz_u(\theta) \\ y_u(r,\theta) = -R\int_0^{2\pi} f(\varphi)\dfrac{\mathrm{d}G}{\mathrm{d}\vec{n}}(R,\varphi;r,\theta)\,\mathrm{d}\varphi \end{cases}$$

where $\vec{n}$ is the unit vector normal to the boundary of D.

2. STATEMENT OF THE OPTIMAL PROBLEM

We attach to the problem (1) - (5) the following quadratic cost functional

$$J(u) = \int_0^{2\pi}\int_0^R\int_0^R K(\theta,r,\rho)y_u(r,\theta)y_u(\rho,\theta)\,\mathrm{d}r\,\mathrm{d}\rho\,\mathrm{d}\theta +$$

$$(7)\qquad +\left\langle G_0 z_u(2\pi), z_u(2\pi)\right\rangle + \int_0^{2\pi}\left\langle G_1(\theta)z_u(\theta), z_u(\theta)\right\rangle \mathrm{d}\theta +$$

$$+\int_0^{2\pi}\left\langle H(\theta)u(\theta)\right\rangle \mathrm{d}\theta$$

where $K(.;.,.):[0,2\pi]\times[0,R]\times[0,R]\to\Re$ is continuous with respect to all variables, $K(\theta;r,\rho) = K(\theta;\rho,r)$ and for any $g \in \mathscr{C}([0,R];\Re)$ we have $\int_0^R\int_0^R K(\theta;r,\rho)g(r)g(\rho)\,\mathrm{d}r\,\mathrm{d}\rho \geq 0$;

$H(.)$ is a symmetric matrix of continuous functions $h_{ij}:[0,2\pi]\to\Re$, of $k\times k$ type and strongly positive definite; G_0 is a constant, symmetric and semipositive definite matrix of $m\times m$ type, $G_1(.)$ is a matrix of integrable functions, of $m\times m$ type with $G_1(\theta)=G_1^*(\theta)$ and $\int_0^{2\pi}\langle G_1(\theta)g(\theta),g(\theta)\rangle d\theta\geq 0$ for any $g\in \mathcal{C}([0,2\pi];\Re)$.

The optimal control problem consists in finding of a function $\tilde{u}\in\mathcal{U}$ such that,

(8) $$J(\tilde{u})=\inf_{u\in\mathcal{U}} J(u).$$

A straightforward calculation shows that $J(.)$ is strictly convex. From the continuous dependence of the $z_u(.)$, of u, one gets that the map $u\in L^2([0,2\pi],\Re^k)\to y_u(.,.)\in\mathcal{Z}$, where $\mathcal{Z}=\{f\mid f:D\to\Re,\ \text{continuous}\}$ endowed with with the topology of the uniform convergence, is continuous so that , $J(.)$ is a continuous functional. Under the hypothesis of the positivity, required for $K(\theta;r,\rho),G_0,G_1(\theta)$ and the assumptions imposed to $H(\theta)$ one gets that $J(.)$ is coercive. Using the existence and uniqueness theorem of the optimal control for coercive, strictly convex and lower semicontinuous ((l.s.c.) in the weak topology) functionals, given in [2] , one gets the existence and uniqueness of the optimal control $u\in\mathcal{U}$, related to the functional (7).

A straightforward calculation expressing the requirement for the derivative of the functional to be zero, (that in the case of the convex functionals is just the necessary and sufficient condition of optimality), leads to the following necessary and sufficient condition that $u\in\mathcal{U}$ should be the optimal control,

(9) $$\begin{aligned}&\int_0^{2\pi}\int_0^R\int_0^R K(\theta;r,\rho)(y_u(\theta,r)-y_{\tilde{u}}(\theta,r))y_{\tilde{u}}(\theta,\rho)dr\ d\rho\ d\theta+\\&+\int_0^{2\pi}\langle H(\theta)\tilde{u}(\theta),u(\theta)-\tilde{u}(\theta)\rangle d\theta+\langle G_0 z_{\tilde{u}}(2\pi),z_u(2\pi)-z_{\tilde{u}}(2\pi)\rangle+\\&+\int_0^{2\pi}\langle G_1(\theta)z_{\tilde{u}}(\theta),z_u(\theta)-z_{\tilde{u}}(\theta)\rangle d\theta=0\end{aligned}$$

3. THE CHARACTERIZATION OF THE OPTIMAL CONTROL

Let us denote by $(p(\theta,r),\psi(\theta))$ the adjoint state, defined as the solution of the following boundary value problem,

(1*) $$\Delta p(\theta,r)=\int_0^{2\pi}K(\theta;r,\rho)y_u(\rho,\theta)d\rho$$

(2*) $$p(R,\theta)=0$$

(4*) $$\frac{d\psi}{d\theta} = -A^*(\theta)\psi + C^* \frac{dp}{d\vec{n}}(R,\theta) + G_1(\theta)z_{\tilde{u}}(\theta)$$

(5*) $$\psi(2\pi) = -G_0 z_{\tilde{u}}(2\pi).$$

Using the adjoint state one gets a characterization of the optimal control, given by,

PROPOSITION 1. *The optimal control $\tilde{u}(.)$ admits the representation*

(10) $$\tilde{u}(\theta) = H^{-1}(\theta)B^*(\theta)\pi(\theta).$$

Proof: Let $\tilde{u}(.)$ be the optimal control for the problem (1) - (5), (7), (8). Then, $\tilde{u}(.)$ satisfies the necessary and sufficient condition of optimality (9). The condition (9) is going to be transformed, applying *Green*'s formula to the couple $(p, y_u - y_{\tilde{u}})$, where y_u, $y_{\tilde{u}}$ are the solutions of the problem (1) - (2), corresponding to the controls u, $\tilde{u}$; $p(\theta, r)$ is the solution of the problem (1*) - (2*). Applying *Green*'s formula to the couple $(p, y_u - y_{\tilde{u}})$ one gets,

$$\int_0^{2\pi} \Big\langle u(\theta) - \tilde{u}(\theta), H(\theta)\tilde{u}(\theta) - B^*(\theta)\Big[\int_{2\pi}^{0} \Phi^*(\theta, s)(C^* \frac{dG}{d\vec{n}}(R,s) + G_2(s)z_{\tilde{u}}(s))\,ds -$$
$$-\Phi^*(\theta, 2\pi)G_0 z_{\tilde{u}}(2\pi) \Big] \Big\rangle d\theta = 0.$$

But the above is true for any $u(.) \in \mathcal{U}$. Therefore, as $u(.)$ ranges over $\mathcal{U}$, $u(.) - \tilde{u}(.)$, ranges over the whole space $\mathcal{U}$, so that $\tilde{u}(\theta) = H^{-1}(\theta)B^*(\theta)\psi(\theta)$, where $\psi(.)$ is the solution of the problem (4*) - (5*).

From the representation (10) for $\tilde{u}(.)$ and the hypothesis that $B(.)$ and $H(.)$ are continuous, we get $\tilde{u}(.) \in \mathcal{C}([0,2\pi]; \Re^k)$.

4. THE HAMILTON TYPE SYSTEM ; THE EXISTENCE AND UNIQUENESS OF THE SOLUTION

We consider the following boundary problem (attached to the *Hamilton* type system).

$$
(11)\quad \begin{cases}
\Delta y(r,\theta) = 0 & \Delta p(r,\theta) = \int_0^{2\pi} K(\theta; r, \rho) y(\rho,\theta)\,\mathrm{d}\rho \\
y(R,\theta) = f(\theta) & p(R,\theta) = 0 \\
f(\theta) = Cz(\theta) & \\
\dfrac{\mathrm{d}z}{\mathrm{d}\theta} = A(\theta)z + & \dfrac{\mathrm{d}\psi}{\mathrm{d}\theta} = -A^*(\theta)\psi + G_1(\theta)z(\theta) + C^* \dfrac{\mathrm{d}p}{\mathrm{d}\bar{n}}(R,\theta) \\
+B(\theta)H^{-1}(\theta)B^*(\theta)\Psi(\theta) & \\
z(0) = z_0 & \psi(2\pi) = -G_0 z(2\pi).
\end{cases}
$$

The previous variational results allows us to prove for the problem (11) the following result:

THEOREM 1.

(i) *There exists a unique solution of the boundary value problem* (11). *This is the quadruple* $(y, z; p, \Psi)$ *corresponding to the optimal control* $\bar{u}(.)$ *related to the problem* (1) - (5), (7), (8).

(ii) *The problem* (11) *can be reduced to a Fredholm integral equation for* $\Psi(.)$.

Proof: See the proof form P.4.1. from **[3]** and use the representation formulas for the solutions of the problems (1) - (5), (1*) - (2*), (4*) - (5*). One gets, for $\psi(.)$, the integral equation

$$(12)\qquad \psi(\theta) = \lambda(\theta) + \int_0^{2\pi} T(\theta,s)\psi(s)\,\mathrm{d}s,$$

where $\lambda(\theta)$ depends only on the initial data z_0 of the problem.(11).

PROPOSITION 2. The *Freedholm* integral equation (12) admits a unique solution of the form,

$$(13)\qquad \psi(\theta) = \lambda(\theta) + \int_0^{2\pi} N(\theta,s)\lambda(s)\,\mathrm{d}s.$$

Proof: See the proof for P.4.2. from **[3]**.

REFERENCES

1. **V. A. Yakubovič** *Frequency - Domain Theorem for the Case when State and Control Spaces are Hilbert Spaces and Applications to Feedback Solution of Some Control Problems.* I. (in Russian) Sibirskii Mat. Journal, XV 3 (1974).
2. **J. L. Lions** *Optimal Control of Systems Governed by Partial Differential Equations*, Springer Verlag, Berlin Heidelberg New York, 1971.
3. **Maria Giurgiu** *A Feedback Solution of a Linear Quadratic Problem for Boundary Control of Heat Equation.* Rev. Roum. Math. Pures et Appl. Tome XX, N° 8, P. 927 - 954, Bucharest 1975.

Received at: 02.09.1996

Department of Mathematics
Military Tehnical Academy
B-dul George Cosbuc nr. 81 - 83, Bucharest
ROMANIA