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ON SOME GENERALIZED CONTRACTIVE TYPE CONDITIONS FOR MULTIVALUED CONDENSING MAPPINGS

Vasile BERINDE

Introduction.

The most convenient ambient space for stating many fixed point theorems for a contraction or a generalized contraction seems to be a *metric space*.However, in this setting - in the absence of the linear structure offered by a Banach space - we can obtain only metrical fixed point theorems.

In order to compensate this drawback Takahashi introduced in 1970 [14] the definition of convexity in metric spaces and generalized some important fixed point theorems previously proved for Banach spaces.

Recently, Gajić and Stojaković [11] obtain a generalization of the Takahashi's result by means of a general contractive type condition. This type of contractivity is expressed by a comparison function, i.e. a real function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ satisfying a few properties of the linear function $\varphi(t)=\alpha t$, $0\leq\alpha<1$, involved in the classical contraction condition

$$d(Tx,Ty) \le \alpha \cdot d(x,y),\ \forall x,y \in X,$$

for a given mapping $T:X \to X$.

A generalized contraction is a mapping $T:X \to X$ satisfying the following generalized condition

$$d(Tx,Ty) \le \varphi(d(x,y)),\ \forall x,y \in X,$$

for a given comparison function φ (for a detailed study of this topic see,for example, Rus, A.I.[16],[17] and Berinde [1]-[10]).

The aim of this paper is to consider alternative conditions for the definition of the comparison function used in [11] and to prove similar theorems for nonexpansive mappings.

2 Convex metric spaces

We need some definitions and remarks from [11],[12] and [14].

Definition 1. Let (X,d) be a metric space and I be the closed unit interval. A mapping $W:X \times X \times I \to X$ is said to be a *convex structure* on X if for all $x,y,u \in X$, $\lambda \in I$,

$$d(u,W(x,y,\lambda)) \le \lambda d(u,x) + (1-\lambda)d(u,y).$$

X endowed with a convex structure is called a (*Takahashi*) *convex metric space*.

Remark. Any convex subset of a normed space is a convex metric space with

$$W(x,y,\lambda) = \lambda x + (1-\lambda)y.$$

Definition 2. Let X be a convex metric space. A nonempty subset K of X is said to be *convex* if

$W(x,y,\lambda) \in K$ whenever $x,y \in K$ and $\lambda \in I$.

Remark. 1) As shown by Takahashi [14],the open and closed balls are convex and the arbitrary intersection of convex sets is a convex set.

2) If we denote for an arbitrary $A\subset X$

$$\tilde{W}(A):=\{W(x,y,\lambda)\,|\,x,y\in A,\ \lambda\in I\},$$

then the convexity may be equivalently defined in a simpler way: K is convex if and only if $\tilde{W}(K)\subset K$.

Definition 3. A convex metric space (X,d) is said to have *Property* (C) if and only if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.

Remark. Every weakly compact convex subset of a Banach space has *Property (C)*.

Definition 4. A subset A of a metric space (X,d) is called *proximal* if for each $x\in X$, there exists an element $a\in A$ such that $d(x,a)=d(x,A)$ ($d(x,A)$ denotes as usually the number $\inf\ \{d(x,y)\,|\,y\in A\}$).

Let's now denote the family of all nonempty bounded proximal subsets of X by 2^X_{BP} and the Haussdorf metric defined on 2^X_{BP} induced by d by H. This means, for $A,B\in 2^X_{BP}$,

$$H(A,B)=\max\{\underset{x\in A}{d}(x,B),\ \underset{x\in B}{d}(x,A)\}.$$

Definition 5. Let $T:X\to 2^X_{BP}$ and $x\in X$. The sequence

$\{x_n/\,x_0=x,\ x_n\in Tx_{n-1}\}$ is called an *orbit of x under T* and is denoted by $\sigma(x)$.

The orbit $\sigma(x)$ is called *strongly regular* if

$\sigma(x)=\{x_n/\,x_n\in Tx_{n-1},\ d(x_n,x_{n-1})=d(x_{n-1},Tx_{n-1})\}$.

Definition 6. A convex metric space (X,d) with the convex structure W is called *P-convex metric space* if for all $x,y,z,u\in X$ and $\lambda\in I$,

$$d(W(x,y,\lambda),W(u,z,\lambda)) \le \lambda d(x,u)+(1-\lambda)d(y,z).$$

Remark. For any subset A of a convex metric space X we have

$$diam\,A = diam(conv\,A),$$

and, if we denote $A_n := \overset{\sim n}{W}(A)$, $A \subset X$, , then we have

$$conv\,A = \lim A_n \left(= \bigcup_{n=1}^{\infty} A_n\right),$$

see [11] and [12].

3. Comparison functions

For details concerning the study of comparison functions and generalized contractions we refer to Rus [16], [17] and Berinde [1]-[10].

Definition 1. A function $\varphi:\mathbb{R}_+ \to \mathbb{R}_+$ is called *comparison* function if it satisfies the following conditions

(i) φ is monotone increasing;

(ii) $(\varphi^n(t))_{n\in N}$ converges to 0, for all $t \ge 0$

(φ^n stands for the n^{th} iterate of φ).

Remark. In the paper [11], Gajić and Stojaković consider a function φ satisfying (i) and the following two conditions:

(iii) φ is right continuous;

(iv) $\varphi(t)<t$ for $t>0$.

Lemma 1 ([17], Lemma 3.1.2). If φ satisfies (i) and (ii) then φ satisfies (iv) and $\varphi(0) = 0$.

Lemma 2 ([17], Lemma 3.1.5). If $\varphi:\mathbb{R}_+ \to \mathbb{R}_+$ satisfies (i) then (iii) is equivalently to the following condition:

(v) φ is right upper semicontinuous.

Lemma 3 ([10], Lemma 1.1.2). If φ is a comparison function then φ is continuous at zero.

Example 1. If $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is given by $\varphi(t)=0$, for $t\in[0,1]$

and $\varphi(t)=\frac{1}{2}t$, for $t>1$, then

1) φ is a comparison function;

2) φ doesn't satisfy condition (iii).

This example suggest us to consider general contractive conditions, similar to those in the paper [11], but with φ a comparison function, i.e.φ satisfying (i) and (ii), instead of (i), (iii) and (iv), as in the quoted paper. We also obtain more general invariance theorems.

4 Invariance theorems for multivalued mappings

The main result of this paper is given by

Theorem 1 *Let (X,d) be a complete P-convex metric space with the continuous convex structure W and K a nonempty closed bounded convex subset of X with Property (C). Let T be a mapping of K into the family of proximal subsets of K satisfying*

$$H(Tx,Ty)\le\varphi(\max\{d(x,Tx),d(y,Ty)\}) \tag{1}$$

for each $x,y\in K$ and φ a given comparison function. Then there exists a nonempty subset M of K such that

$$T(M)\subseteq M.$$

Proof. For any $x\in K$, we may construct a strongly regular orbit at x_0 under T,

$$\sigma(x_0)=\{x_n/x_n\in T_{n-1},\ n\ge1\}.$$

We shall prove that $\lim\limits_{n\to\infty} d(x_n,Tx_n)=0$. (2)

Let us denote $d(x_n,Tx_n)$ by C_n .Then,from the contraction condition (1) we have

$$C_n=d(x_n,Tx_n)\le H(Tx_{n-1},Tx_n)\le\varphi(\max\{d(x_{n-1},Tx_{n-1}),\ d(x_n,Tx_n)\}. \tag{3}$$

If $C_{n-1}<C_n$, then from (2) we obtain

$$C_n\le\varphi(C_n)$$

and from property (iv) of a comparison function we deduce

$$C_n \le \varphi(C_n) < C_n,$$

contradiction.

Hence $C_{n-1} \ge C_n$ and from (3) we obtain

$$C_n \le \varphi(C_{n-1}) < C_{n-1},$$

that is (C_n) is a monotone decreasing sequence of nonnegative real numbers.This means

$$\lim_{n\to\infty} C_n = C$$

exists and C≥0. From (i) and

$$C_n \le \varphi(C_{n-1})$$

we deduce by induction

$$C_{n+1} \le \varphi^n(C_1)$$

So, from

$$0 \le C_{n+1} \le \varphi^n(C_1)$$

and by condition (ii) we obtain $C = \lim_{n\to\infty} C_n = 0$.

Although the remainder proof is essential identical to the one in [11] we repeat it here for convenience.

For each $\epsilon>0$ let us denote

$$H_\varepsilon = \{ x / d(x, Tx) \le \epsilon \}$$

and $A_\varepsilon = T(H_\varepsilon)$.

From (2) we have that $H_\varepsilon \ne \phi$ for each $\epsilon>0$. We shall prove that

$$\overline{conv}\ A_\varepsilon \subseteq H_\varepsilon, \quad \text{for each} \quad \epsilon > 0 .$$

Let $y \in A_\varepsilon$ and let $\delta > 0$ be given.Then there exists $y' \in conv\ A_\varepsilon$ such that

$$d(y, y') \le \delta .$$

Since $y' \in conv\ A_\varepsilon$ there exists $n_o \in \mathbb{N}$ such that

$$y' \in \overset{-n_o}{W}(A_\varepsilon) .$$

This means

$y^* \in W(y_1, y_2, \lambda)$, with $y_1, y_1 \in \tilde{W}^{n_0-1}(A_\varepsilon)$ and $\lambda_1 \in I$

and further $y_1 = W(y_{11}, y_{12}, \lambda_{11})$, $y_2 = W(y_{21}, y_{22}, \lambda_{12})$,

$\lambda_{11}, \lambda_{12} \in I, y_{11}, y_{12}, y_{21}, y_{22} \in \tilde{W}^{n_0-2}(A_\varepsilon)$ and so on.

After no more than n_0 steps we shall obtain elements belonging to A_ε .Let denote them by $\{y_i^*\}_{i \in I_1}$ (Obviously I_1 is a finite set).

Since $y_i^* \in A_\varepsilon$, $i \in I_1$, there exists $y_i \in H_\varepsilon$ such that $y_i^* \in T(y_i)$.

But Ty is proximal, hence, for every $i \in I_1$, for some $z_i \in Ty$, we have

$$d(y_i^*, z_i) = d(y_i^*, Ty).$$

Now let z be defined by $\{z_i\}_{i \in I_1}$ in the same way as y^* by $\{y_i^*\}_{i \in I_1}$.

Obviously $z \in Ty$. On an other hand

$$d(y, Ty) \le d(y, y^*) + d(y^*, Ty) \le \delta + d(W(y_1, y_2, \lambda_1), W(z_1, z_2, \lambda_1)) \le$$

$$\le \delta + \lambda_1 d(y_1, z_1) + (1-\lambda_1) d(y_2, z_2) \le \delta + \lambda_1 d(W(y_{11}, y_{12}, \lambda_{12}),$$

$$W(z_{11}, z_{12}, \lambda_{12})) + (1-\lambda_1) d(W(y_{21}, y_{22}, \lambda_{21}), W(z_{21}, z_{22}, \lambda_{12})) \le$$

$$\le \delta + \lambda_1 \cdot \lambda_{12} d(y_{11}, z_{11}) + \lambda_1 (1-\lambda_{12}) d(y_{12}, z_{12}) + (1-\lambda_1) \lambda_{21} d(y_{21}, z_{21}) +$$

$$+ (1-\lambda_1)(1-\lambda_2) d(y_{22}, z_{22}) \le \ldots \le$$

$$\le \delta + \sum_{i \in J} \alpha_i d(y_i^*, z_i) = \delta + \sum_{i \in J} \alpha_i d(y_i^*, Ty) \le$$

$$\le \delta + \sum_{i \in J} \alpha_i H(Ty_i, Ty) \le \delta + \sum_{i \in J} \alpha_i \varphi(\max\{ d(y_i, Ty_i), d(y, Ty)\} \le$$

$$\le \delta + \sum_{i \in J} \alpha_i \varphi(\max(\varepsilon, d(y, Ty)),$$

where $\alpha_i \ge 0, i \in J$ and $\sum_{i \in J} \alpha_i = 1$ (J is a finite set).

If $d(y,Ty)>\epsilon$, than

$$d(y,Ty)\le\delta+\varphi(d(y,Ty)).$$

Since $\delta>0$ is arbitrary and φ satisfies (iv) this leads to a contradiction

$$d(y,Ty)\le\varphi(d(y,Ty))<d(y,Ty).$$

Hence we must have $d(y,Ty)\le\epsilon$, i.e. $y\in H_\epsilon$.

This proves $\overline{conv}\ T(H_\epsilon)\subseteq H_\epsilon$.

Let $B=\{\overline{conv}\ T(H_\epsilon)/\epsilon>0\}$. Then B is a bounded decreasing net of nonempty closed convex subsets and in view of property(C) it has nonempty intersection:

$$\emptyset\ne\bigcap B\subseteq\bigcap\{H_\epsilon/\epsilon>0\}.$$

This shows that the function $x\to d(x,Tx)$ attains its infimum over K and due to (2) this infimum must be zero.
Now let take $M=\bigcap\{H_\epsilon/\epsilon>0\}$ and the proof is complete.

Remark. Using the proofs of Theorem 1 and Theorem 2 [14] and the same arguments as in [11] one can prove:

Theorem 2. *Let (X,d) be a complete P-convex metric space with continuous convex structure and K a nonempty convex closed bounded subset of X with Property (C). Let T be a mapping of K into the family of nonempty proximal subsets of K which satisfies the condition:*
for given $\epsilon>0$, there exists $\delta>0$ such that for any $x,y\in K$

$$\epsilon\le\max\{d(x,Tx),d(y,Ty)\}\le\epsilon+\delta\to H(Tx,Ty)<\epsilon.$$

Then there exists a nonempty subset M of K such that $T(M)\subseteq M$.

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Universitatea din Baia-Mare
Facultatea de Litere şI Ştiinţe
str. Victoriei nr.76
RO-4800 Baia-Mare
ROMANIA