Buletinul Ştiinţific al Universităţii din Baia Mare

Seria B, Matematică-Informatică, vol.XII (1996),191-202

Dedicated to the 35th anniversary of the University of Baia Mare

ON SOME CONTINUOUS RUNGE - KUTTA METHODS

Iulian COROIAN

Abstract. Continuous half-explicit Runge - Kutta methods for differential-algebraic system of index 2 are considered.
A family of continuous half-explicit Runge-Kutta methods of uniform order $p=4$ and $s=6$ stages is derived.

KEY WORDS: differential-algebraic systems, Runge-Kutta methods, continuous extension.
AMS subject classification: 65 L05, 65 L06

1. INTRODUCTION

Many real problems from mechanics, physics, engineering etc. can be modeled by initial value problems for differential-algebraic system of the form

(1.1.a) $$y'(x) = f(y(x), z(x)),$$

(1.1.b) $$0 = g(y(x)),$$

(1.2.) $$y(x_o) = y_o,\ z(x_o) = z_o,$$

where $f:\mathbb{R}^{m+k}\to\mathbb{R}^m$, $g:\mathbb{R}^m\to\mathbb{R}^k$, $x_o\in[a,b]\subset\mathbb{R}$, $y_o\in\mathbb{R}^m$, $z_o\in\mathbb{R}^k$.

We assume that the vector functions $f=(f_1,f_2,\dots,f_m)$ and $g=(g_1,g_2,\dots,g_k)$ are nonlinear and they are sufficiently smooth such that the matrices

$$g_y'(y)=\left(\frac{\partial g_i}{\partial y^j}\right),\ f_z'(y,z)=\left(\frac{\partial f_j}{\partial z^i}\right);\quad i=\overline{1,k};\ j=\overline{1,m},$$

are cotinuous and the consistency conditions

(1.3) $$g(y_o)=0,\quad g_y'(y_o)\cdot f_z'(y_o,z_o)=0,$$

are satisfied. It is also assumed that in a neighbourhood of the solution of (1.1) - (1.2), there exists the bounded inverse

$$\left[g'_y(y) \cdot f'_z(y,z) \right]^{-1},$$

so the problem (1.1) - (1.2) has index 2.
An example of problems of the form (1.1) - (1.2) is the multibody system with constraints on the velocity level, see [8]. Also, the differential equations with discontinuities in the right side, leads to systems of the form (1.1). Much works has been devoted to development of numerical methods for problem (1.1)-(1.2), especially implicit Runge - Kutta type methods, [5], [7], [9] and half-explicit Runge - Kutta methods, [1]. In the last years many authors derived so called continuous Runge - Kutta mathods for numerical solution of initial value problems for differential systems, [3], [4], [5], [7], [10], [11], [12], [14].
The present work is dedicated to extension the applicability of these continuous methods to the numerical solution of the problem (1.1) - (1.2) of index 2.

2. PRELIMINARY RESULTS

If we use the works of Hairer, Lubich and Roche, [9], and Brasey an Harier [1], we can give

DEFINITION 2.1 A continuous half-explicit Runge - Kutta type methods with s stages for the $y(x)$ - component of the solution of the problem (1.1) - (1.2) provides a continuous approximation $u(x)$ (or an interpolant $u(x)$) for the exact solution $y(x)$, by using a uniform mesh of $[a,b]$,

$$\{a=x_0<x_1<\ldots<x_N=b\},\ x_{i+1}-x_i=h,\ i=\overline{1,N-1},$$

and the following relations

$$Y_{ni}=y_n+h\sum_{j=1}^{i-1} a_{ij}f(Y_{nj}Z_{nj}) \qquad (2.1)$$

$$0=g(Y_{ni}),\ i=1,2,\ldots,s, \qquad (2.2)$$

$$u(x_n+\theta h)=y_n+h\sum_{i=1}^{s} b_i(\theta)\, f(Y_{ni},Z_{ni}) \qquad (2.3)$$

$$0=g(u(x_n+\theta h)),\ \theta\in[0,1],\ n=0,1,2,\ldots, \qquad (2.4)$$

where $b_i(\theta)$, $i=\overline{1,s}$ are polynominals of degree at most p, p being the order of the method and $b_i(0)=0$, $i=\overline{1,6}$; a_{ij}, $i=\overline{2,s}$, $j=\overline{1,i-1}$ are real parameters. The value y_n, $n=0,1,2,\ldots$ are the solution of local problem given by (1.1) and $y(x_n)=y_n$. We consider like in (1.3) for $n=0$ that $g(y_n)=0$.

DEFINITION 2.2 The half-explicit continuous Runge - Kutta method defined by (2.1) - (2.4) has the **uniform order** p, if p is the larges integer such that

$$\max_{0\le\theta\le 1}\|y(x_n+\theta h)-u(x_n+\theta h)\|=O(h^{p+1}),$$

where $y(x)$ is the y- component solution of (1.1) - (1.2), satisfying the local condition

$$y(x_n)=u(x_n)=y_n,\ n=0,1,2,\ldots$$

and $\|\cdot\|$ is any norm on $\mathbb{R}^m$.

REMARK 2.1 It in known that a discret Runge - Kutta method, that is b_i are constants, the order of the method (called nodal order) is greater or equal to the uniform order of the corresponding to continuous Runge - Kutta method.

REMARK 2.2 The coefficients a_{ij} of the half-explicit method (2.1) -(2.2) form a strictly inferior triangular matrix

$$A=\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0\\ a_{21} & 0 & 0 & \ldots & 0 & 0\\ a_{31} & a_{32} & 0 & \ldots & 0 & 0\\ . & . & . & \ldots & . & .\\ a_{s1} & a_{s2} & a_{s3} & \ldots & a_{s,s-1} & 0 \end{pmatrix},$$

and we shall require

$$c_1=0,\ c_i=\sum_{j=1}^{i-1}a_{ij},\quad i=2,3,\ldots,s-1,\ \ c_s=1$$

Then, the half-explicit continuous Runge - Kutta method, can be writen in an array

$$\begin{array}{c|c} c & A \\ \hline & b^T, \end{array}$$

with $c=(c_1,c_2,\ldots,c_s)^T$ and $b^T(\theta)=(b_1(\theta),b_2(\theta),\ldots,b_s(\theta))$.

REMARK 2.3. For effectiv application of the half-explicit continuous method (2.1)-(2.4) we proceed as follows: for $i=1$ from (2.1) we get $Y_{n1}=y_n$ and (2.2) will be satisfied. If $i=2$, we obtain Y_{n2} from (2.1) and insert it in (2.2), we get a nonlinear equation for Z_1

$$g(y_n+ha_{21}f(Y_{n1},Z_{n1}))=0 \quad .$$

With Z_{n1} computed, (2.1) give us Y_{n2} explicitly. We repeat this procedure in next stages. In the i-1- th stage for the computation of $Z_{n,i-1}$ we have the system

$$g\left(y_n+h\sum_{j=1}^{i-1} a_{ij}f(Y_{nj},Z_{nj}) \right)=0$$

This system must be solved approximatively, and then we obtain Y_{ni} explicitly from (2.1).

REMARK 2.4. The continuous half-explicit methods defined by (2.1) - (2.4) are assumed to exist and they are convergent. Results in this respect for the discret implicit methods and half-explicit methods can be found in [9] and [1], respectively.

3. ORDER CONDITIONS FOR CONTINUOUS METHODS.

We will require that the coefficients a_{ij}, $i=\overline{2,s}$; $j=\overline{1,i-1}$, c_i, $i=\overline{1,s}$ and the weighted polynomials $b_i(\theta)$, $i=\overline{1,s}$ satisfy conditions to ensure that the local error be of a certain order. These order conditions are obtained with the aid of Taylor expansions, in power of $h=x_{i+1}-x_i$,of exact solution $y(x_n+\theta h)$ and of the approximate solution $u(x_n+\theta h)$.

The coeficients of these expansions are functions defined on a set of rooted trees. We will not enter into details, as all the theory can be found in [2] and [9]. We will only present the order conditions for the continuous mrthod provided by (2.1) - (2.4) to have the uniform order $p=4$.

If we assume that $a_{i,i-1}\neq 0,\ i=\overline{2,s+1}$ with $a_{s+1,i}=b_i(\theta)$ and $c_{s+1}=1$ then the matrix

$$A=\begin{pmatrix} a_{21} & 0 & 0 & \cdots & 0 & 0 \\ a_{31} & a_{32} & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{s1} & a_{s2} & a_{s3} & \cdots & a_{s,s-1} & 0 \\ a_{s+1,1} & a_{s+1,2} & a_{s+1,3} & \cdots & a_{s+1,s-1} & a_{s+1,s} \end{pmatrix}$$

is invertible and we note its invers as

$$A^{-1}=(\omega_{ij})=\begin{pmatrix} \omega_{11} & 0 & 0 & \cdots & 0 \\ \omega_{21} & \omega_{22} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \omega_{s1} & \omega_{s2} & \cdots & \cdots & \omega_{ss} \end{pmatrix},$$

with $\omega_{ii}\neq 0,\ i=\overline{1,s}$.

We wish to underline the fact that if the continuous method (2.1) -(2.4) has the uniform order p and s stages then it is necessary that $s\geq 2p-2$ if $c_1=0,\ c_i\neq 0,\ i=\overline{2,s}$, [4]. For $p=4$ it is necessary that $s\geq 6$, that is we will take $s=6$, therefore the minimal number of stages for order $p=4$. The order conditions for $p=4$ are

(3.1) $$\sum_i b_i(\theta)=\theta,$$

(3.2) $$\sum_{i} b_i(\theta)\, c_i = \frac{\theta^2}{2},$$

(3.3) $$\sum_{i} b_i(\theta)\, c_i^2 = \frac{\theta^3}{3},$$

(3.4) $$\sum_{i,j} b_i(\theta)\, a_{ij} c_j = \frac{\theta^3}{6},$$

(3.5) $$\sum_{i,j} b_i(\theta)\, c_i \omega_{ij} c_{j+1}^2 = \frac{2}{3}\theta^3,$$

(3.6) $$\sum_{i,j,k} b_i(\theta)\, \omega_{ij} c_{j+1}^2 \omega_{ik} c_{k+1}^2 = \frac{4}{3}\theta^3,$$

(3.7) $$\sum_{i} b_i(\theta)\, c_i^3 = \frac{\theta^4}{4},$$

(3.8) $$\sum_{i,j} b_i(\theta)\, c_i a_{ij} c_j = \frac{\theta^4}{8},$$

(3.9) $$\sum_{i,j} b_i(\theta)\, a_{ij} c_j^2 = \frac{\theta^4}{12},$$

(3.10) $$\sum_{i,j,k} b_i(\theta)\, a_{ij} a_{jk} c_k = \frac{\theta^4}{24},$$

(3.11) $$\sum_{i,j} b_i(\theta)\, c_i^2 \omega_{ij} c_{j+1}^2 = \frac{\theta^4}{2},$$

(3.12) $$\sum_{i,j,k} b_i(\theta)\, c_i \omega_{ij} c_{j+1}^2 \omega_{ik} c_{k+1}^2 = \theta^4,$$

(3.13) $$\sum_{i,j,k,l} b_i(\theta)\, \omega_{ij} c_{j+1}^2 \omega_{ik} c_{k+1}^2 \omega_{il} c_{l+1}^2 = 2\theta^4,$$

(3.14) $$\sum_{i,j} b_i(\theta)\, c_i \omega_{ij} c_{j+1}^3 = \frac{3}{4}\theta^4,$$

(3.15) $$\sum_{i,j,k} b_i(\theta)\, c_i \omega_{ij} c_{j+1} a_{j+1,k} c_k = \frac{3}{8}\theta^4,$$

(3.16) $$\sum_{i,j} b_i(\theta)\, a_{ij} c_j \omega_{ij} c_{j+1}^2 = \frac{\theta^4}{4},$$

(3.17) $$\sum_{i,j,k} b_i(\theta)\, \omega_{ij} c_{j+1}^2 \omega_{ik} c_{k+1}^3 = \frac{3}{2}\theta^4,$$

(3.18) $$\sum_{i,j,k,l} b_i(\theta)\, \omega_{ij} c_{j+1}^2 \omega_{ik} c_{k+1} a_{k+1,l} c_l = \frac{3}{4}\theta^4,$$

(3.19) $$\sum_{i,j,k} b_i(\theta)\, a_{ij} c_j \omega_{jk} c_{k+1}^2 = \frac{\theta^4}{6},$$

$$\sum_{i,j,k,l} b_i(\theta)\, a_{ij}\omega_{jk}c_{k+1}^2\omega_{jl}c_{l+1}^2 = \frac{\theta^4}{3} \tag{3.20}$$

In order to derive a half-explicit continuous Runge - Kutta method (2.1) - (2.4) we have therefore to determine a_{ij}, c_i, $b_i(\theta)$ to satisfy the system (3.1) - (3.20).
At first site the solution of this algebraic nonlinear system seems to be a utopia; we will nevertheless notice that it can be considerably simplified under certain circumstances.

4. SIMPLIFICATION OF OEDER CONDITIONS

PROPOSITION 4.1 If $b_2(\theta) \equiv 0$ and a_{ij}, c_i, $i=\overline{2,s}; j=\overline{1,i-1}$ satisfy the relations

$$\sum_{j=1}^{i-1} a_{ij}c_j = \frac{1}{2}c_i^2, \quad i=3,4,5,6, \tag{4.1}$$

then we also have

$$\sum_{j=1}^{i} \omega_{ij}c_{j+1}^2 = 2c_i + \omega_{i1}c_2^2, \quad i=\overline{1,6}. \tag{4.2}$$

The proof goes the same as in [1], we will therefore not go into further details.

PROPOSITION 4.2 If $b_2(\theta) \equiv 0$ and moreover

$$\sum_{j=1}^{i-1} a_{ij}c_j^2 = \frac{1}{3}c_i^3, \quad i=3,4,5,6, \tag{4.3}$$

then we also have

$$\sum_{j=1}^{i} \omega_{ij}c_{j+1}^3 = 3c_i^2 + \omega_{i1}c_2^3, \quad i=\overline{1,6} \tag{4.4}$$

Proof. The relations (4.3) can be written as

$$A\begin{pmatrix} c_1^2 \\ c_2^2 \\ \cdot \\ \cdot \\ \cdot \\ c_5^2 \\ c_6^2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 \\ c_3^3 \\ c_4^3 \\ \cdot \\ \cdot \\ \cdot \\ c_6^3 \\ c_7^3 \end{pmatrix}, \quad c_7=1$$

Hence by multiplication with A^{-1} on the left, we will get

$$3I\begin{pmatrix} c_1^2 \\ c_2^2 \\ \cdot \\ \cdot \\ \cdot \\ c_6^2 \end{pmatrix} = \bar{A}^{-1}\begin{pmatrix} 0 \\ c_3^3 \\ c_4^3 \\ c_5^3 \\ c_6^3 \\ c_7^3 \end{pmatrix},$$

where I is the identity matrix of order 6. This relation can be written

$$\sum_{j=1}^{i} \omega_{ij} c_{j-1}^3 = 3c_i^2, \quad i=2,3,4,5,6$$

If we add $\omega_{i1}c_2^3$ to the two members of the above ralation we will get the very relation (4.4).

PROPOSITION 4.3 If $b_2(\theta)\equiv 0$, (4.1) and (4.2) hold, and moreover $b_i(\theta)\omega_{i1}=0$, $i=\overline{1,6};\theta\in[0,1]$ then the system (3.1) - (3.20) is reduced to the equations (3.1), (3.2), (3.3) (3.7), that is to

(4.5) $$b_3(\theta)+b_4(\theta)+b_5(\theta)+b_6(\theta)=\theta,$$

(4.6) $$b_3(\theta)c_3+b_4(\theta)c_4+b_5(\theta)c_5+b_6(\theta)c_6=\frac{\theta^2}{2},$$

(4.7) $$b_3(\theta)c_3^2+b_4(\theta)c_4^2+b_5(\theta)c_5^2+b_6(\theta)c_6^2=\frac{\theta^3}{3},$$

(4.8) $$b_3(\theta)c_3^3+b_4(\theta)c_4^3+b_5(\theta)c_5^3+b_6(\theta)c_6^3=\frac{\theta^4}{4}.$$

Proof. Let us notice that the equation (3.4) is satisfied on the hypothesis that (3.1), (3.2), (3.3) hold

$$\sum_{i,j} b_i(\theta)\, a_{ij} c_j = \sum_i b_i(\theta) \cdot \sum_j a_{ij} c_j = \sum_{i=1}^{6} b_i(\theta)\, \frac{c_i^2}{2} = \frac{1}{2} \sum_{i=1}^{6} b_i(\theta)\, c_i^2 = \frac{1}{2}\, \frac{\theta^3}{3} = \frac{\theta^3}{6}.$$

Here we have made use of the relation (4.1) valid for $i=3,4,5,6$; as for $i=1$ we have $c_1=0$ and for $i=2$ we have $b_2(\theta)=0$.
Likewise for the equation (3.5) we have

$$\sum_{i,j} b_i(\theta)\, c_i \omega_{ij} c_{j+1}^2 = \sum_i b_i(\theta)\, c_i \sum_j \omega_{ij} c_{j+1}^2 = \sum_{i=1}^{6} b_i(\theta)\, c_i (2c_i + \omega_{i1} c_2^2) =$$

$$2\sum_{i=1}^{6} b_i(\theta)\, c_i^2 + \sum_{i=1}^{6} b_i(\theta)\, \omega_{i1} c_i c_2^2 = 2 \cdot \frac{\theta^3}{3} + 0 = \frac{2}{3}\theta^3,$$

In a similar way we can prive that the equations (3.6), (3.8)-(3.20) are satisfied in our hypothesis.
For example for the equation (3.14) we have, on the basis of proposition 4.2.

$$\sum_{i,j} b_i(\theta)\, c_i \omega_{ij} c_{j+1}^3 = \sum_i b_i(\theta)\, c_i \sum_j \omega_{ij} c_{j+1}^3 = \sum_i b_i(\theta)\, c_i (3c_i^3 + \omega_{i1} c_2^3) =$$

$$= 3\sum_{i=1}^{6} b_i(\theta)\, c_i^3 + c_2^3 \sum_{i=1}^{6} b_i(\theta)\, \omega_{i1} c_i = 3\,\frac{\theta^4}{4}.$$

REMARK 4.1. Given the fact that $\omega_{11} a_{21} = 1$, we have $\omega_{11} \neq 0$ and by hypothesis $b_i(\theta)\,\omega_{i1} = 0$, $i=\overline{1,6}$, we get $b_1(\theta) \equiv 0$.

5. THE EFFECTIVE DERIVATION OF A FAMILY OF HALF-EXPLICIT METHODS

COROLLARY 5.1. In order to obtain a continuous half-explicit method of order 4 with 6 stages it suffices that we determine the parameters of the method a_{ij}, c_i, $b_i(\theta)$ so as to have the equations (4.5) - (4.8) satisfied and, in addition,

(5.1) $$c_1 = 0,\; b_1(\theta) \equiv 0,\; b_2(\theta) \equiv 0,$$

(5.2) $$\sum_{j=1}^{i-1} a_{ij} = c_i, \quad i=\overline{2,6},$$

(5.3) $$\sum_{j=1}^{i-1} a_{ij} c_j = \frac{c_i^2}{2}, \quad i=\overline{3,6},$$

(5.4) $$\sum_{j=1}^{i-1} a_{ij} c_j^2 = \frac{1}{3} c_i^3, \quad i=\overline{3,6},$$

(5.5) $$\omega_{31} = \omega_{41} = \omega_{51} = \omega_{61} = 0$$

The proof of the corolary results from propositions 4.1, 4.2, 4.3.

PROPOSITION 5.1. If c_3, c_4, c_5 are real distinct numbers and differ from 1 then the polynomials $b_i(\theta)$, $i=\overline{3,6}$ which satisfy the system (4.5) - (4.8) are given by the relations

(5.6)

$$b_3(\theta)=\frac{-1}{(c_4-c_3)(c_5-c_3)(1-c_3)}\left[\frac{\theta^4}{4}-\frac{\theta^3}{3}(1+c_4+c_5)+\frac{\theta^2}{2}(c_4+c_4+c_4c_5)-c_4c_5\theta\right],$$

(5.7)

$$b_4(\theta)=\frac{1}{(c_4-c_3)(c_5-c_4)(1-c_4)}\left[\frac{\theta^4}{4}-\frac{\theta^3}{3}(1+c_3+c_5)+\frac{\theta^2}{2}(c_3+c_5+c_3c_5)-c_3c_5\theta\right],$$

(5.8)

$$b_5(\theta)=\frac{-1}{(c_5-c_3)(c_5-c_4)(1-c_5)}\left[\frac{\theta^4}{4}-\frac{\theta^3}{3}(1+c_3+c_4)+\frac{\theta^2}{2}(c_3+c_4+c_3c_4)-c_3c_4\theta\right],$$

(5.9)

$$b_6(\theta)=$$

$$=\frac{1}{(1-c_3)(1-c_4)(1-c_5)}\left[\frac{\theta^4}{4}-\frac{\theta^3}{3}(c_3+c_4+c_5)+\frac{\theta^2}{2}(c_3c_4+c_3c_5+c_4c_5)-c_3c_4c_5\theta\right].$$

Proof. Assuming that c_3, c_4, c_5 are distinct and differ from 1, the linear system in unknowns $b_i(\theta)$ (4.5) - (4.8) has its determinant different from 0 and then the system has a unique solution which can be found easily and we will get (5.6) - (5.9).

PROPOSITION 5.2. A family of solutions depending on one parameter for the system (5.2) - (5.4) is

$$c_1=0,\ a_{21}=c_2,\ a_{31}=\frac{3}{8}c_2,\ a_{32}=\frac{9}{8}c_2,\ c_3=\frac{3}{2}c_2,$$

$$a_{41}=a_{42}=0,\ a_{43}=\frac{9}{4}c_2,\ c_4=\frac{9}{4}c_2,$$

$$a_{51}=a_{52}=0,\ a_{53}=\frac{c_5(9c_2-2c_5)}{3c_2},\ a_{54}=\frac{2c_5(c_5-3c_2)}{3c_2},$$

(5.10)

$$c_5=\frac{9(7+\sqrt{17})}{8}c_2,\ c_6=1,\ a_{61}=a_{62}=0,$$

$$a_{63}=\frac{18c_2c_5-9c_2-12c_5+8}{9c_2(3c_2-2c_5)},\ a_{64}=\frac{4(18c_2c_5-9c_2-6c_5+4)}{9c_2(4c_5-9c_2)},$$

$$a_{65}=\frac{81c_2^2-72c_2c_5+24c_5-9c_2-8}{3(2c_5-3c_2)(4c_5-9c_2)},$$

where $c_2\in\mathbb{R}\setminus\{0,1\}$ represents the parameter.

The conclusion is arrived at as a consequence of very elaborate computation from the relations (5.2) - (5.5), forming for each and every $i\in\{2,3,4,5,6\}$ a linear system in a_{ij} whose solution lead us to the values (5.10).

COROLLARY 5.2. The coefficients a_{ij}, c_i, $i=\overline{2,6}$, $j=\overline{1,i-1}$ given by (5.10) together with the weighted polynomials $b_i(\theta)$, given by (5.6)-(5.9) where c_3, c_4, c_5 have the values in (5.10) and $b_1(\theta)=0$, $b_2(\theta)=0$, will provide a family of continuous half-explicit Runge - Kutta type methods having the uniform order $p=4$ and $s=6$ stages.
The proof of the statement is immediate taking into consideration the fact that the system (3.1)-(3.20) can be reduced to the system (5.6)-(5.9) in the hypotheses (5.1)-(5.5).

COROLLARY 5.3. A particular continuous half explicit method of order 4 with 6 stages obtained by the choice $c_2=\frac{1}{6}$ is given in the array

$$
\text{(5.11)}\qquad
\begin{array}{c|cccccc}
0 & & & & & & \\
\frac{1}{6} & \frac{1}{6} & & & & & \\
\frac{1}{4} & \frac{1}{16} & \frac{3}{16} & & & & \\
\frac{3}{8} & 0 & 0 & \frac{3}{8} & & & \\
\frac{3(7+\sqrt{17})}{16} & 0 & 0 & -\frac{9(7+\sqrt{17})}{32} & \frac{3(7+\sqrt{17})}{32} & & \\
1 & 0 & 0 & \frac{1+3\sqrt{17}}{6} & \frac{19-11\sqrt{17}}{18} & \frac{-2+\sqrt{17}}{9} & \\
\hline
 & 0 & 0 & b_3(\theta) & b_4(\theta) & b_5(\theta) & b_6(\theta)
\end{array}
$$

$$
\text{(5.12)}\qquad
\begin{aligned}
b_3(\theta) &= \frac{-17+3\sqrt{17}}{51}\left[16\theta^4-\frac{4}{3}(43+3\sqrt{17})\theta^3+\frac{3(93+\sqrt{17})}{4}\theta^2-\frac{9(7+\sqrt{17})}{2}\theta\right],\\
b_4(\theta) &= \frac{5-\sqrt{17}}{15}\left[32\theta^4-\frac{8(41+3\sqrt{17})}{3}\theta^3+(121+15\sqrt{17})\theta-6(7+\sqrt{17})\theta\right],\\
b_5(\theta) &= \frac{-289+71\sqrt{17}}{51}\left[\theta^4-\frac{13}{6}\theta^2+\frac{23}{16}\theta^3-\frac{3}{8}\theta\right],\\
b_6(\theta) &= \frac{5-3\sqrt{17}}{15}\left[\theta^4-\frac{31+3\sqrt{17}}{12}\theta^3+\frac{3(39+5\sqrt{17})}{64}\theta^2-\frac{9(7+\sqrt{17})}{128}\theta\right]
\end{aligned}
$$

REMARKS 5.1. The continuous half-explicit method given by (5.11) - (5.12) has the property that

$$\text{(5.13)}\qquad b'_{i+3}(\theta)=L_i(\theta),\quad i=1,2,3,$$

where $L_i(\theta)$ is the i-th Lagrange elementary interpolation polynomial with respect to the set of the nodes $\{c_3, c_4, c_5, c_6\} = \left\{\frac{1}{4}, \frac{3}{8}, \frac{3(7+\sqrt{17})}{16}, 1\right\}$. corresponding to non-zero weights. The statement can be checked easily.

REFERENCES

1. **BRASEY V., HAIRER E**, Half-explicit Runge - Kutta methods for differential-algebraic systems of index 2, SIAM. J. Numer. Anal., 202(1993), 538-512
2. **BUTHER J.C.**, The numerical analysis of ordinary differential equations, Wiley, Chichester (1987)
3. **CALVO M, MONTIJANO J.I., RÁNDEZ L**, A fifth order interpolant for the Dormand and Prince Runge - Kutta method, J. Comp. Appl. Math., 29 (1990) pag. 91-100
4. **CARNICER J.M.**, A lower bound for the number of stages of an explicit continuous Runge - Kutta methods to obtain convergence of given order, BIT, 31(1991), pag.364-368
5. **DORMAND J.R., PRINCE, P.J.**, A family of embedded Runge - Kutta formulae, J. Comp. Appl. Math., 6(1980), pag. 19-26
6. **DORMAND, J.R., PRINCE, P.J.** Runge - Kutta triples, Comput. Math. Appl. 12A (1986), pp. 1007-1017
7. **DORMAND J.R., PRINCE, P.J.**, Practical Runge - Kutta processes, SIAM J. Sci. Statist. Comput., 5(1989), pp. 977-989.
8. **FÜHRER C, LEIMKUHLER, B.**, Numerical solution of differential-algebraic equations for constrained mechanical motion, Numer. Math., 59(1991), 55-69.
9. **HAIRER E, LUBICH CH., ROCHE M**, The Numerical Solution of Differential-Algebraic System by Runge - Kutta Methods, Springer Lecture Notes in Math. 1409, Springer Verlag, Berlin- New-York (1989)
10. **MUIR P., OWREN B.**, Order barriers and characterizations for continuous mono-implicit Runge - Kutta schemes, Math. Comp. 61, 204(1993), 675-699
11. **OWREN B., ZENNARO M.**, Order barriers for continuous explicit Runge - Kutta methods, Math. Comp. 56.(1991), 645-661
12. **SHAMPINE L.F.**, Interpolation for Runge - Kutta methods, SIAM J. Numer. Anal., 22(1985), 1014-1027
13. **SHAMPINE L.F.**, Some practical Runge - Kutta formulas, Math.Comp., 173(1986), 135-150
14. **VERNER J.**, Differentiable interpolants for high-order Runge - Kutta methods, SIAM J. Numer.Anal. 30, 5(1993) 1446-1466
15. **ZENNARO M.**, Natural continuous extensions of Runge - Kutta methods, Math. Comp. 46(1986), pp. 119-133

Received at: 02.09.1996

University of Baia Mare
Str. Victoriei nr. 76
RO-4800 Baia Mare
ROMANIA