Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol.XII (1996),203-210

Dedicated to the 35th anniversary of the University of Baia Mare

COMPARISON THEOREMS FOR THE SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. Our aim in this paper is to establish new comparison principles of Sturm's type. The equation

$$(r(t)u'(t))' + p(t)|u(g(t))|^{\alpha} \operatorname{sgn} u(g(t)) = 0 \tag{*}$$

is compared with the equation

$$(l(t)y'(t))' + z(t)|y(w(t))|^{\beta} \operatorname{sgn} y(w(t)) = 0.$$

We consider the second order differential equations with deviating argument

$$\big(r(t)u'(t)\big)' + p(t)|u(g(t))|^{\alpha} \operatorname{sgn} u(g(t)) = 0, \tag{1}$$

and

$$\big(l(t)y'(t)\big)' + z(t)|y(w(t))|^{\beta} \operatorname{sgn} y(w(t)) = 0, \tag{2}$$

where r, l, p, z, g, $w \in C([t_0, \infty))$ are positive, $\alpha \geq \beta > 0$, and $g(t) \to \infty$ as $t \to \infty$ and $w(t) \to \infty$ as $t \to \infty$.

In the sequel we shall restrict our attention to nontrivial solutions of the equations considered. Such a solution is called oscillatory if the set of its zeros is unbounded. Otherwise, it is said to be nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

We say that equation (1) is in canonical form if

$$\int^{\infty} \frac{ds}{r(s)} = \infty.$$

On the other hand if

$$\int^{\infty} \frac{ds}{r(s)} < \infty$$

then equation (1) is said to be in noncanonical form.

It is known that the canonical equation (1) is oscillatory if $\int^{\infty} p(s)\, ds = \infty$, hence dealing with equation (1) which is in canonical form we may assume that $\int^{\infty} p(s)\, ds < \infty$.

1991 *Mathematics Subject Classification.* Primary 34C10.
Key words and phrases. Canonical and noncanonical equations, comparison theorem.

We make use the following functions in the remainder of this paper:

$$R(t) = \int_{t_0}^{t} \frac{ds}{r(s)}, \quad \text{and} \quad L(t) = \int_{t_0}^{t} \frac{ds}{l(s)}, \quad t \geq t_0$$

for canonical case of (1) and (2) and

$$\rho(t) = \int_{t}^{\infty} \frac{ds}{r(s)}, \quad \text{and} \quad \lambda(t) = \int_{t}^{\infty} \frac{ds}{l(s)}, \quad t \geq t_0$$

for noncanonical case of (1) and (2).

Most of the work in the theory of oscillations is centered around the second order differential equations. There is much literature dealing with this subject, and for a systematic treatment the reader is referred to [6].

Since Sturm (1836) introduce the concept of oscillation when he studied the problem of heat transmission, oscillation theory has been an important area of research in the theory of differential equations. The prototype of results we wish to establish is the following Sturm's classical comparison theorem [9], which deals with the special cases of equations (1) and (2), namely with the equations

$$(r(t)u'(t))' + p(t)u(t) = 0 \tag{3}$$

and

$$(l(t)y'(t))' + z(t)y(t) = 0. \tag{4}$$

Theorem A. *Assume that*

$$r(t) \leq l(t), \quad t \in [t_0, \infty), \tag{5}$$

$$p(t) \geq z(t), \quad t \in [t_0, \infty). \tag{6}$$

Then equation (3) is oscillatory if equation (4) is oscillatory.

Using Sturm's comparison theorem, we can obtain the oscillatory property of an second order differential equation from some other equation with known oscillatory behavior. In fact, many good oscillation criteria have been obtained from Sturm's comparison theorem. For example taking the fact that the Euler equation $y''(t) + (a/t^2)y(t) = 0$ is oscillatory if $a > 1/4$ into account, according to Theorem A equation (3) is oscillatory if $t^2 p(t) > (1+\epsilon)/4$, for some $\epsilon > 0$.

The following analogy of Sturm's comparison theorem for nonlinear differential equations with deviating argument has been given in [2].

Theorem B. *Let (1) and (2) be in canonical form. Assume that (5) holds and the following conditions are satisfied:*

$$g(t) \geq w(t), \quad t \geq t_0, \tag{7}$$

$$w \text{ is nondecreasing}, \tag{8}$$

$$\int_t^{\infty} p(s)\, ds \geq \int_t^{\infty} z(s)\, ds, \quad t \geq t_0. \tag{9}$$

Then equation (1) is oscillatory if equation (2) is oscillatory.

Note that condition (5) imposed on the functions r and l in Theorems A and B has the effect that those theorems cannot be used if (1) is in noncanonical form and (2) is in canonical form.

Another question concerning the oscillation of equations of the form (3) and (4) may be posed as follows. Suppose that one of the inequalities (5) or (6) is violated, but the other is amply satisfied. It is possible to deduce again that (3) is oscillatory? This question has been considered by Harris [4].

The main tool in our efforts here will be a transformation of an equation. We recall the following two results from [3] and [7]. Note that the function ρ which belongs to the class $C^1([t_0,\infty))$, is decreasing and maps the interval $[t_0,\infty)$ onto the interval $(0,\rho_0]$, where $\rho_0 = \rho(t_0)$. Let ρ^{-1} be the inverse function to ρ. Then a composite function $\rho^{-1}(1/s)$ belongs to the class $C^1([s_0,\infty))$, where $s_0 = 1/\rho_0$, is increasing on this interval and maps the interval $[s_0,\infty)$ onto $[t_0,\infty)$.

Let us denote

$$p_1(s) = \frac{1}{s^3}\, p\left(\rho^{-1}(1/s)\right)\, r\left(\rho^{-1}(1/s)\right)\, \rho^{\alpha}\left(g\left[\rho^{-1}(1/s)\right]\right), \quad s \ge s_0, \tag{10}$$

$$g_1(s) = \frac{1}{\rho\left(g\left[\rho^{-1}(1/s)\right]\right)}, \quad s \ge s_0. \tag{11}$$

Then we have the following:

Theorem C. *Let (1) be in noncanonical form. Then equation (1) is oscillatory if and only if the equation*

$$y''(t) + p_1(t)|y[g_1(t)]|^{\alpha} \operatorname{sgn} y[g_1(t)] = 0, \quad t \ge 1/\rho(t_0) \tag{12}$$

is oscillatory.

Theorem C can be found in [3, Corollary 1]. Moreover, Corollary 1 in [3] provides the relationship between a solution of (1) and a solution of (12). For more details see [3].

Now we turn to the canonical case of (1). Let us denote

$$p_2(t) = r\left(R^{-1}(t)\right) p\left(R^{-1}(t)\right), \quad t \ge t_0,$$
$$g_2(t) = R\left(g\left(R^{-1}(t)\right)\right), \quad t \ge t_0,$$

where R^{-1} is the function which is the inverse of R. Then the following theorem which can be found in [7] holds.

Theorem D. *Let (1) be in canonical form. Then equation (1) is oscillatory if and only if the equation*

$$y''(t) + p_2(t)|y[g_2(t)]|^{\alpha} \operatorname{sgn} y[g_2(t)] = 0, \tag{13}$$

is oscillatory.

Now we are prepared to present some new comparison results. For simplicity and further references let us denote for noncanonical case of (2)

$$z_1(s) = \frac{1}{s^3}\, z\left(\lambda^{-1}(1/s)\right)\, l\left(\lambda^{-1}(1/s)\right)\, \lambda^{\beta}\left(w\left[\lambda^{-1}(1/s)\right]\right), \quad s \ge s_0, \tag{14}$$

$$w_1(s) = \frac{1}{\lambda\left(w\left[\lambda^{-1}(1/s)\right]\right)}, \quad s \ge s_0, \tag{15}$$

and for canonical case of (2) we define

$$z_2(t) = l\left(L^{-1}(t)\right) z\left(L^{-1}(t)\right), \quad t \geq t_0,$$
$$w_2(t) = L\left(w\left(L^{-1}(t)\right)\right), \quad t \geq t_0.$$

Then we have:

Theorem 1. *Let (1) and (2) be in canonical form. Assume that (8) is satisfied. Further assume that for all large t*

$$g_2(t) \geq w_2(t), \qquad \text{and}$$

$$\liminf_{t\to\infty} R^{\epsilon}(t) \int_t^\infty p(s)\,ds > \limsup_{t\to\infty} L^{\epsilon}(t) \int_t^\infty z(s)\,ds, \tag{16}$$

for some $\epsilon > 0$. Then equation (1) is oscillatory if equation (2) is oscillatory.

Proof. Since (8) holds the function w_2 is nondecreasing. By Theorem D, equation (1) is oscillatory if and only if equation (13) is oscillatory and since equation (2) is oscillatory then equation

$$y''(t) + z_2(t)|y[w_2(t)]|^{\beta} \operatorname{sgn} y[w_2(t)] = 0, \tag{17}$$

is oscillatory. Applying Theorem B to equations (13) and (17) we see that equation (13) is oscillatory if

$$\int_t^\infty p_2(s)\,ds \geq \int_t^\infty z_2(s)\,ds,$$

for all large t, say $t \geq t_1$. Simple computation shows that the last inequality is equivalent to

$$\int_{R^{-1}(t)}^\infty p(s)\,ds \geq \int_{L^{-1}(t)}^\infty z(s)\,ds, \quad t \geq t_1. \tag{18}$$

On the other hand, as $R(t) \to \infty$ if and only if $t \to \infty$, we have from (16)

$$\liminf_{t\to\infty} t^{\epsilon} \int_{R^{-1}(t)}^\infty p(s)\,ds > \limsup_{t\to\infty} t^{\epsilon} \int_{L^{-1}(t)}^\infty z(s)\,ds, \tag{19}$$

from which it follows (18). The proof is complete.

Theorem 2. *Let (1) and (2) be in noncanonical form. Assume that (8) is satisfied. Further assume that for all large t*

$$g_1(t) \geq w_1(t), \qquad \text{and}$$

$$\liminf_{t\to\infty} \frac{1}{\rho^{\epsilon}(t)} \int_t^\infty \rho(s)\rho^{\alpha}\left(g(s)\right)p(s)\,ds > \limsup_{t\to\infty} \frac{1}{\lambda^{\epsilon}(t)} \int_t^\infty \lambda(s)\lambda^{\beta}\left(w(s)\right)z(s)\,ds, \tag{20}$$

for some $\epsilon > 0$. Then equation (1) is oscillatory if equation (2) is oscillatory.

Proof. It is easy to see that function w_1 is nondecreasing. According to Theorem C, equation (1) is oscillatory if and only if equation (12) is oscillatory. On the other hand since equation (2) is oscillatory then equation

$$y''(t) + z_1(t)|y[w_1(t)]|^\beta \operatorname{sgn} y[w_1(t)] = 0 \tag{21}$$

is oscillatory. Theorem B applied to equations (21) and (12) guarantees the oscillation of (12) if

$$\int_t^\infty p_1(s)\,ds \geq \int_t^\infty z_1(s)\,ds,$$

for all large t. It is easy to verify that the last inequality is equivalent to

$$\int_{\rho^{-1}(1/t)}^\infty \rho(s)\rho^\alpha\big(g(s)\big)p(s)\,ds \geq \int_{\lambda^{-1}(1/t)}^\infty \lambda(s)\lambda^\beta\big(w(s)\big)z(s)\,ds. \tag{22}$$

On the other hand, as $\rho(1/t) \to \infty$ if and only if $t \to \infty$, the assumption (20) is equivalent to

$$\liminf_{t\to\infty} t^\epsilon \int_{\rho^{-1}(1/t)}^\infty \rho(s)\rho^\alpha\big(g(s)\big)p(s)\,ds > \limsup_{t\to\infty} t^\epsilon \int_{\lambda^{-1}(1/t)}^\infty \lambda(s)\lambda^\beta\big(w(s)\big)z(s)\,ds. \tag{23}$$

from which it follows (22). The proof is complete.

Theorem 3. *Let (1) be in canonical form and (2) be in noncanonical form. Assume that (8) is satisfied. Further assume that for all large t*

$$g_2(t) \geq w_1(t) \qquad \text{and}$$

$$\liminf_{t\to\infty} R^\epsilon(t)\int_t^\infty p(s)\,ds > \limsup_{t\to\infty} \frac{1}{\lambda^\epsilon(t)}\int_t^\infty \lambda(s)\lambda^\beta\big(w(s)\big)z(s)\,ds,$$

for some $\epsilon > 0$. Then equation (1) is oscillatory if equation (2) is oscillatory.

Theorem 4. *Let (1) be in noncanonical form and (2) be in canonical form. Assume that (8) is satisfied. Further assume that for all large t*

$$g_1(t) \geq w_2(t) \qquad \text{and}$$

$$\liminf_{t\to\infty} \frac{1}{\rho^\epsilon(t)}\int_t^\infty \rho(s)\rho^\alpha\big(g(s)\big)p(s)\,ds > \limsup_{t\to\infty} L^\epsilon(t)\int_t^\infty z(s)\,ds,$$

for some $\epsilon > 0$. Then equation (1) is oscillatory if equation (2) is oscillatory.

Theorem 3 and Theorem 4 can be proved similarly as Theorem 1 and 2 and so the proofs of those theorems can be omitted.

Theorem 4 permit us to deduce oscillation of noncanonical equation from that of canonical equation. This is a new fact, which nor Sturm's comparison theorem nor other known comparison theorems provide. Moreover Theorems 1-4 enable us to deduce oscillation of (1) from a given oscillatory equation. If we return to the problem from the motivation part of this paper we see that if one of the inequalities (5) or (6) $\big((5)$ or $(9)\big)$is violated but the other is amply satisfied, then by Theorems 1-4 we can again deduce oscillation of (3) from (4) $\big($of (1) from (2)$\big)$. Our results here extend those of Harris [4].

Example 1. Let us consider the Euler equation

$$\left(\frac{1}{t}x'(t)\right)' + \frac{a}{t^3}x(t) = 0, \quad t \geq 1, \quad a > 0. \tag{24}$$

It is easy to verify, that (24) is oscillatory if $a > 1$. Put $\epsilon = 1$ in (16), then by Theorem 1 applied to (3) and (24) we obtain that canonical equation (3) is oscillatory if

$$\liminf_{t\to\infty} R(t)\int_t^\infty p(s)\,ds > \frac{1}{4},$$

which is a generalization of a result of Chanturia [1]. Moreover Theorem 1 provides another oscillation criterion for (3), in fact taking (16) into account we see that (3) is oscillatory if

$$\liminf_{t\to\infty} R^\epsilon(t)\int_t^\infty p(s)\,ds > 0,$$

for some $\epsilon \in (0,1)$. Now we turn to noncanonical case of (3). By Theorem 2 noncanonical equation (3) is oscillatory if

$$\liminf_{t\to\infty} \frac{1}{\rho(t)}\int_t^\infty \rho^2(s)p(s)\,ds > \frac{1}{4} \quad \text{or} \quad \liminf_{t\to\infty} \frac{1}{\rho^\epsilon(t)}\int_t^\infty \rho^2(s)p(s)\,ds > 0,$$

for some $\epsilon \in (0,1)$.

As the following illustrative example shows Theorems 1-4 can be used to find a sufficient condition under which equation (1) is not oscillatory, i.e. (1) has a nonoscillatory solution.

Example 2. Consider the nonlinear differential equation

$$\left(t^3y'(t)\right)' + t^3y^3(t) = 0, \quad t \geq 1. \tag{25}$$

Equation (25) is not oscillatory since it has a nonoscillatory solution $y(t) = t^{-1}$. Consider another nonlinear differential equation

$$\left(l(t)u'(t)\right)' + z(t)u^3(w(t)) = 0, \tag{26}$$

where l, z and w are the same as in (2) and further $w(t) \leq t$ is nondecreasing. Then by Theorem 4 with $\epsilon = 2$ canonical equation (26) has a nonoscillatory solution if

$$\limsup_{t\to\infty} L^2(t)\int_t^\infty z(s)\,ds < \frac{1}{16}$$

and by Theorem 2 noncanonical equation (26) has a nonoscillatory solution if

$$\limsup_{t\to\infty} \frac{1}{\lambda^2(t)}\int_t^\infty \lambda^4(s)z(s)\,ds < \frac{1}{16}.$$

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Received at: 02.09.1992

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