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EXISTENCE AND PROPERTIES OF NONOSCILLATORY SOLUTIONS OF THE n-TH ORDER DIFFERENTIAL EQUATION

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Abstract

Necessary and sufficient conditions for a n-th order differential equation of the form

$$\left(p_{n-1}(t)\left(p_{n-2}(t)\left(\ldots(p_1(t)\varphi(y'(t)))'\ldots\right)'\right)'\right)'+f(t,y(h_1(t)),\ldots,y(h_m(t)))=0,$$

are given such that the equation has nonoscillatory solutions with special properties.

Key words: nonlinear differential equation with quasiderivatives, nonoscillatory solutions.

We consider the n-th order differential equation of the form

$$L_ny(t)+f(t,y(h_1(t)),\ldots,y(h_m(t)))=0, \tag{1}$$

where

$$\left.\begin{aligned} &L_1y(t)=p_1(t)\varphi(y'(t)),\\ &L_iy(t)=p_i(t)(L_{i-1}y(t))', \qquad \text{for } i=2,\ldots,n-1,\\ &L_ny(t)=(L_{n-1}y(t))', \end{aligned}\right\} \tag{2}$$

p_i, h_j: $[0,\infty)\to \mathrm{R}$, φ: $\mathrm{R}\to\mathrm{R}$, f: $[0,\infty)\times \mathrm{R}^n\to\mathrm{R}$ are continuous functions, $p_i(t)>0$, $h_j(t)\to\infty$ for $t\to\infty$, $i=1,\ldots,n-1$, $j=1,\ldots,m$, $\varphi(\mathrm{R})=\mathrm{R}$, $\operatorname{sgn}\varphi(u)=\operatorname{sgn}u$ and if $y_1,\ldots,y_m$ are positive (or negative) then $f(t,y_1,\ldots,y_m)$ is also positive (or negative) and nondecreasing in $y_1,\ldots,y_m$.

Throughout the paper we shall assume that

$$\int_0^\infty \frac{1}{p_i(t)}dt=\infty \qquad \text{for } i=2,\ldots,n-1 \tag{3}$$

and

$$\int_0^\infty \left|\varphi^{-1}\left(\frac{k}{p_1(t)}\right)\right|dt=\infty \qquad \text{for all constant } k\neq 0, \tag{4}$$

where φ^{-1}: $\mathrm{R}\to\mathrm{R}$ denotes inverse function of φ. By a solution of equation (1) is meant a function $y(t)$, such that $L_iy(t)$, $1\le i\le n$ exist and are continuous on $[T_y,\infty)$ and

$y(t)$ satisfies (1). We restrict our considerations to those solutions of (1) which exist on some ray $[T_y, \infty)$ and satisfy

$$\sup\{|y(t)| : t_1 \le t < \infty\} > 0 \quad \text{for any } t_1 \in [T_y, \infty).$$

LEMMA 1. *If $y(t)$ is nonoscillatory solution of equation (1), then there exist an integer l, $0 \le l \le n$ and $t_1 \ge t_0$ with $n + l$ odd such that*

$$\left.\begin{aligned} &y(t)L_i y(t) > 0, \quad 1 \le i \le l, \\ &(-1)^{i-l} y(t) L_i y(t) > 0, \quad l \le i \le n, \end{aligned}\right\} \tag{5}$$

for all $t \ge t_1$,

$$\lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |L_i y(t)| = \infty, \quad \textit{for } i = 1, \dots, l-2,$$

$$\lim_{t\to\infty} L_{l-1} y(t) \ne 0, \quad \lim_{t\to\infty} L_n y(t) \quad \textit{is own}$$

and

$$\lim_{t\to\infty} L_j y(t) = 0, \quad \textit{for } j = l+1, \dots, n-1.$$

Lemma generalizes a well-known lemma of Kiguradze and can be proved similarly.

A function $y(t)$ satisfying (5) is said to be a function of degree l (see [2]). The set of all nonoscillatory solutions of degree l od (1) is denoted by $\mathcal{N}_l$.

If

$$\mathcal{N}_l^1 = \{y \in \mathcal{N}_l : \lim_{t\to\infty} L_l y(t) \ne 0\}, \qquad \mathcal{N}_l^0 = \{y \in \mathcal{N}_l : \lim_{t\to\infty} L_l y(t) = 0\},$$

then $\mathcal{N}_l = \mathcal{N}_l^1 \cup \mathcal{N}_l^0$,

$$\mathcal{N}_0 = \mathcal{N}_2 = \ldots = \mathcal{N}_{2k} = \emptyset \qquad \text{for } n \text{ even}$$

and

$$\mathcal{N}_1 = \mathcal{N}_3 = \ldots = \mathcal{N}_{2k+1} = \emptyset \qquad \text{for } n \text{ odd}.$$

We shall use the following notation

$$\left.\begin{aligned} &\Phi_{k,T}(p_1, \dots, p_j; t) = \int_T^t \varphi^{-1}\left(\frac{1}{p_1(s_1)} \int_T^{s_1} \frac{1}{p_2(s_2)} \cdots \right. \\ &\left. \cdots \int_T^{s_{j-1}} \frac{k}{p_l(s_j)} \mathrm{d}s_j \ldots \mathrm{d}s_2 \right) ds_1, \\ &\Phi_k(p_1, \dots, p_j; t) = \Phi_{k,0}(p_1, \dots, p_j; t), \qquad \text{for } j = 1, 2, \dots n-1. \end{aligned}\right\} \tag{6}$$

Using (3), (4) we have that

$$\left.\begin{aligned} &\lim_{t\to\infty} |\Phi_{k,T}(p_1, \dots, p_l; t)| = \infty \quad \text{for all } k \ne 0, \\ &|\Phi_{k,T}(p_1, \dots, p_l; t)| > |\Phi_{l,T}(p_1, \dots, p_l; t)| \quad \text{for all } |k| > |l|,\ kl > 0, \\ &\Phi_{k,T}(p_1, \dots, p_l; T) = 0. \end{aligned}\right\} \tag{7}$$

THEOREM 1. *Let equation* (1) *has a nonoscillatory solution* $y \in \mathcal{N}_l^1$. *Then*

$$\int_0^\infty |f(t, c\Phi_k(p_1, \dots, p_l; h_1(t)), \dots, c\Phi_k(p_1, \dots, p_l; h_m(t)))| \, dt < \infty \tag{8}$$

for some constants $k \neq 0$ *a* $c > 0$.

PROOF. Let $y(t)$ be a nonoscillatory solution of equation (1) belonging $\mathcal{N}_l^1$. We may suppose that $y(t) > 0$ for $t \geq t_0 \geq T_y$ (the proof for $y(t) < 0$ is similar). It follows from Lemma 1 that

$$L_1 y(t) > 0, \dots, L_l y(t) y(t) > 0, \ , L_{l+1} y(t) < 0,$$

$$\lim_{t \to \infty} L_l y(t) = c_1 > 0 \quad \text{and} \quad L_l y(t) > c_1 \quad \text{for every } t \geq t_1 \geq t_0.$$

We can easily find out that

$$y(t) \geq \Phi_{c_1, t_1}(p_1, \dots, p_l; t)$$

so them exists a positive constant c_2 such that

$$y(t) \geq c_2 \Phi_{c_1}(p_1, \dots, p_l; t) \quad \text{for every } t \geq t_1 \geq t_0. \tag{9}$$

Integrating (1) from t to ∞ and using properties of the solution $y(t)$ we obtain

$$\int_t^\infty f(s, y(h_1(s)), \dots, y(h_m(s))) \, ds < \infty.$$

From the last inequality in view of (9) and nondecreasing of $f(s, y_1, \dots, y_m)$ we obtain

$$\int_t^\infty f(s, c_2 \Phi_{c_1}(p_1, \dots, p_l; h_1(s)), \dots, c_2 \Phi_{c_1}(p_1, \dots, p_l; h_m(s))) \, ds < \infty,$$

which implies (8) for $c_1 > 0$ a $c_2 > 0$. This completes the proof.

THEOREM 2. *Suppose that for each fixed* $k \neq 0$ *and* $T \geq 0$

$$\lim_{l \to 0, lk > 0} \frac{\Phi_{l,T}(p_1, \dots, p_{n-1}; t)}{\Phi_{k,T}(p_1, \dots, p_{n-1}; t)} = 0 \tag{10}$$

uniformly on any interval of the form $[T', \infty)$, $T' > T$. *Equation* (1) *has a nonoscillatory solution* $y(t) \in \mathcal{N}_{n-1}^1$ *if* (8) *holds for* $l = n - 1$ *and some* $k' \neq 0$ *and* $c > 0$.

PROOF. Suppose that (8) holds for some $c > 0$ and $k' > 0$. As (10) holds, there exists l_1, $k > l_1 > 0$, such that

$$\Phi_{l_1}(p_1, \dots, p_{n-1}; t) < c\Phi_{k'}(p_1, \dots, p_{n-1}; t).$$

Let $T > 0$, $l > 0$ be such that $2l < l_1 < k$ and

$$\int_T^\infty f(t, \Phi_{2l}(p_1, \dots, p_{n-1}; h_1(t)), \dots, \Phi_{2l}(p_1, \dots, p_{n-1}; h_m(t))) \, dt < l. \tag{11}$$

Let $C[T,\infty)$ be the space of all continuous functions which are defined on $[T,\infty)$ with topology of uniform convergence on compact subintervals of $[T,\infty)$. We define the set

$$Y=\{y\in C[T,\infty):\ \Phi_{l,T}(p_1,\dots,p_{n-1};t)\le y(t)\le \Phi_{2l,T}(p_1,\dots,p_{n-1};t),\ \ t\ge T\} \quad (12)$$

and the mapping $\mathcal{F}:Y\to C[T,\infty)$ by formula

$$\mathcal{F}y(t)=\int_T^t \varphi^{-1}\left(\frac{1}{p_1(s_1)}\int_T^{s_1}\frac{1}{p_2(s_2)}\int_T^{s_2}\cdots\int_T^{s_{n-2}}\frac{1}{p_{n-1}(s_{n-1})}\Big(l+\right. \quad (13)$$

$$\left.+\int_{s_{n-1}}^{\infty} f\left(s,y\left(h_1(s)\right),\dots,y\left(h_m(s)\right)\right)ds\right)ds_{n-1}\dots ds_2\Big)ds_1,$$

for every $t\ge T$. We will prove that the mapping $\mathcal{F}$ fulfils the assumptions of the Schauder-Tychonoff fixed point theorem.

(i) $\mathcal{F}$ *maps* Y *into itself*, because if $y\in Y$, then for every $\tau\ge T$ we have

$$l\le l+\int_\tau^\infty f\left(s,y\left(h_1(s)\right),\dots,y\left(h_m(s)\right)\right)ds\le$$

$$\le l+\int_T^\infty f\left(s,\Phi_{2l}\left(p_1,\dots,p_{n-1};h_1(s)\right),\dots,\Phi_{2l}\left(p_1,\dots,p_{n-1};h_m(s)\right)\right)ds\le 2l,$$

$$\int_T^t \varphi^{-1}\left(\frac{1}{p_1(s_1)}\int_T^{s_1}\frac{1}{p_2(s_2)}\int_T^{s_2}\cdots\int_T^{s_{n-2}}\frac{l}{p_{n-1}(s_{n-1})}ds_{n-1}\dots ds_2\right)ds_1\le$$

$$\le \mathcal{F}y(t)\le$$

$$\le\int_T^t \varphi^{-1}\left(\frac{1}{p_1(s_1)}\int_T^{s_1}\frac{1}{p_2(s_2)}\int_T^{s_2}\cdots\int_T^{s_{n-2}}\frac{2l}{p_{n-1}(s_{n-1})}ds_{n-1}\dots ds_2\right)ds_1$$

The last relation implies that $\mathcal{F}y\in Y$.

(ii) $\mathcal{F}$ *is continuous*. Let y_n be a sequence of elements of Y converging to y in topology of $C[T,\infty)$. The Lebesgue dominates convergence theorem shows that

$$\int_T^\infty f\left(t,y_n\left(h_1(t)\right),\dots,y_n\left(h_m(t)\right)\right)dt\to\int_T^\infty f\left(t,y\left(h_1(t)\right),\dots,y\left(h_m(t)\right)\right)dt\quad\text{for } n\to\infty$$

and so

$$\int_t^\infty f\left(s,y_n\left(h_1(s)\right),\dots,y_n\left(h_m(s)\right)\right)ds\to\int_t^\infty f\left(s,y\left(h_1(s)\right),\dots,y\left(h_m(s)\right)\right)ds\quad\text{for } n\to\infty$$

uniformly on $[T,\infty)$. It follows that $\mathcal{F}y_n(t)\to\mathcal{F}y(t)$ uniformly on compact subintervals of $[T,\infty)$, which implies convergence $\mathcal{F}y_n(t)\to\mathcal{F}y(t)$ in $[T,\infty)$.

(iii) $\mathcal{F}(Y)$ *is relatively compact*. For any $y\in Y$ we get the relation

$$0\le(\mathcal{F}y)'(t)=\varphi^{-1}\Big(\frac{1}{p_1(t)}\int_T^t\frac{1}{p_2(s_2)}\int_T^{s_2}\cdots\int_T^{s_{n-2}}\frac{1}{p_{n-1}(s_{n-1})}\,(l+$$

$$+\int_{s_{n-1}}^{\infty} f\left(s, y\left(h_1(s)\right), \ldots, y\left(h_m(s)\right)\right) ds \Bigg) ds_{n-1} \ldots ds_2 \Big) \le$$

$$\le \varphi^{-1}\Big(\frac{1}{p_1(t)}\int_T^t \frac{1}{p_2(s_2)}\int_T^{s_2} \cdots \int_T^{s_{n-2}} \frac{1}{p_{n-1}(s_{n-1})}\Bigg(l+\int_{s_{n-1}}^{\infty} f\left(s, \Phi_{2l}\left(p_1, \ldots, p_{n-1}; h_1(s)\right), \ldots\right.$$

$$\left.\ldots, \Phi_{2l}\left(p_1, \ldots, p_{n-1}; h_m(s)\right)\right) ds\Big) ds_{n-1} \ldots ds_2\Big)$$

Therefore, applying the Schauder-Tychonoff fixed point theorem we see that there exists an element $y \in Y$ such that $y = \mathcal{F}y$. Consequently

$$y(t) = \int_T^t \varphi^{-1}\left(\frac{1}{p_1(s_1)}\int_T^{s_1} \frac{1}{p_2(s_2)}\int_T^{s_2} \cdots \int_T^{s_{n-2}} \frac{1}{p_{n-1}(s_{n-1})}\Big(l+\right.$$

$$\left.+\int_{s_{n-1}}^{\infty} f\Big(s, y\left(h_1(s)\right), \ldots, y\left(h_m(s)\right)\Big) ds\Big) ds_{n-1} \ldots ds_2\right) ds_1.$$

Differentiating the last integral equation we obtain that $y(t)$ is a solution of equation (1) and $y(t) \in \mathcal{N}_{n-1}^1$.

THEOREM 3 *Let the assumption of Theorem 2 hold and $l = n-1$. Then equation (1) has a nonoscillatory solution $y(t) \in \mathcal{N}_{n-1}^1$ if and only if (8) holds for some constants $k' \neq 0$ and $c > 0$.*

Theorem 3 follows from Theorem 1 and Theorem 2.

THEOREM 4. *Equation (1) has a nonoscillatory solution $y(t) \in \mathcal{N}_0^1$ if and only if*

$$\int_0^{\infty}\left|\varphi^{-1}\left(\frac{-1}{p_1(t)}\int_t^{\infty} \frac{1}{p_2(s_2)}\int_{s_2}^{\infty} \cdots \int_{s_{n-1}}^{\infty} f(s, c, \ldots, c) ds ds_{n-1} \ldots ds_2\right)\right| dt < \infty \tag{14}$$

for some constant $c \neq 0$.

PROOF. Let $y(t)$ be a positive solution of equation (1) belonging $\mathcal{N}_0^1$ and let $\lim_{t\to\infty} y(t) = c_1 > 0$. From the equation (1) using properties of solution $y(t)$ we have

$$y(t) - y(t_0) = \int_{t_0}^t \varphi^{-1}\left(\frac{-1}{p_1(s_1)}\int_{s_1}^{\infty} \frac{1}{p_2(s_2)}\int_{s_2}^{\infty} \cdots\right. \tag{15}$$

$$\left.\cdots\int_{s_{n-1}}^{\infty} f\left(s, y\left(h_1(s)\right), \ldots, y\left(h_m(s)\right)\right) ds ds_{n-1} \ldots ds_2\right) ds_1$$

Because $c_2 \ge y\left(h_i(t)\right) \ge c_1$, $c_2 \ge y(t) \ge c_1$ and $f(t, y_1, \ldots, y_m)$ is nondecreasing in $y_1, \ldots, y_m$, we obtain

$$-f(t, c_2, \ldots, c_2) \le -f(t, y\left(h_1(s)\right), \ldots, y\left(h_m(s)\right)) \le -f(t, c_1, \ldots, c_1). \tag{16}$$

From (15) and (16) it is obvious that

$$0 > \int_{t_0}^{\infty} \varphi^{-1}\left(\frac{-1}{p_1(t)} \int_t^{\infty} \frac{1}{p_2(s_2)} \int_{s_2}^{\infty} \cdots \int_{s_{n-1}}^{\infty} f(s, c_1, \ldots, c_1) ds ds_{n-1} \ldots ds_2\right) dt > -\infty,$$

We have proved that the relation (14) holds for c_1.

Now suppose that (14) holds for some $c > 0$. Then there exists a $T > 0$ such that

$$0 > \int_{T}^{\infty} \varphi^{-1}\left(\frac{-1}{p_1(t)} \int_t^{\infty} \frac{1}{p_2(s_2)} \int_{s_2}^{\infty} \cdots \int_{s_{n-1}}^{\infty} f(s, c, \ldots, c) ds ds_{n-1} \ldots ds_2\right) dt > -\frac{c}{2}.$$

Define the set

$$Y = \{y \in C[T, \infty) : \ \frac{c}{2} \leq y(t) \leq c, \ t \geq T\}$$

where $C[T, \infty)$ is the topological space as in the proof of Theorem 2 and the mapping

$$\mathcal{F}y(t) = c + \int_T^t \varphi^{-1}\left(\frac{-1}{p_1(s_1)} \int_{s_1}^{\infty} \frac{1}{p_2(s_2)} \int_{s_2}^{\infty} \cdots \right.$$

$$\left. \cdots \int_{s_{n-1}}^{\infty} f\left(s, y\left(h_1(s)\right), \ldots, y\left(h_m(s)\right)\right) ds ds_{n-1} \ldots ds_2\right) ds_1,$$

for every $t \geq T$. Proceeding as in the proof of Theorem 2 we get, that for the mapping $\mathcal{F}$ the assumptions of the Schauder-Tychonoff fixed point theorem hold which implies the existence of the continuous function $y \in Y$ such that

$$y(t) = c + \int_T^t \varphi^{-1}\left(\frac{-1}{p_1(s_1)} \int_{s_1}^{\infty} \frac{1}{p_2(s_2)} \int_{s_2}^{\infty} \cdots \right.$$

$$\left. \cdots \int_{s_{n-2}}^{\infty} \frac{1}{p_{n-2}(s_{n-2})} \int_{s_{n-1}}^{\infty} f\left(s, y\left(h_1(s)\right), \ldots, y\left(h_m(s)\right)\right) ds ds_{n-2} \ldots ds_2\right) ds_1$$

Consequentive differentiation of the last relation shows that $y(t)$ is the solution of equation (1) belonging $\mathcal{N}_0^1$. This completes the proof.

REMARK 1. *Theorems generalize some results from the paper [1] and [3]. Exactly if $n = 3$, $m = 1$, $h_1(t) = t$ we obtain Theorems 1,2,3,4 from the paper [3] and if $n = 2$, $m = 1$, $h_1(t) = t$ we obtain Theorems 1 from the paper [1].*

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