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BOUNDARY INVERSE PROBLEMS IN THE CASE OF INCOMPRESSIBLE FLUID JETS

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Abstract: *The paper deals with the "aposteriori" design of a symmetric airfoil for a problem involving a two-dimensional incompressible fluid jet, by prescribing "apriori"the velocity along the contour. This is the inverse boundary value problem and a singular integral equation is derived for the velocity angle. For prescribed velocity distributions, the geometrical equations of the airfoilare established numerically and the drag coefficient Cx is also computed.*

I. Basic Equations

Let consider the plane, potential steady flow of a perfect, incompressible fluid, without external body forces (Hyp). In an usual formulation, [1], [3], there are considered in the physical plane D_z, $z = x + iy$, the complex potential $f(z) = \varphi(x, y) + i\psi(x, y)$ and the complex velocity $w(z) = u(x, y) + iv(x, y)$, where $\varphi = \varphi(x, y)$ is the velocity potential and $\psi = \psi(x, y)$ the stream function; u and v are the velocity components $\vec{V} = u\vec{i} + v\vec{j}$; θ is the angle of the velocity with the x - *axis*. From $\frac{df}{dz} = \bar{w} = u - iv$ and $w = Ve^{i\theta}$ we have the relation, [1], [3]:

$$d\varphi + id\psi = (u - iv)\,(dx + idy). \tag{1}$$

In this framework, let consider two-dimensional flow between the parallel plates AD and $A'D'$, where the upstream unperturbed velocity (in (A, A')) is $\vec{V}_1 = V_1\vec{i}$. A symmetric curvilinear obstacle $(BO'B)$ is placed in the flow field.

The free streamlines (jet lines) BC and $B'C'$ separe from the points B and B' of the obstacle. On these lines the values of the velocity, pressure and density are $V°$, $p°$, $\rho°$ respectively; at the infinity downstream, in the points (C, D) and (C', D') the velocity is $\vec{V}° = V°\vec{i}$, where the symmetry axis A_0O of the figure is $x'Ox$. Because of the symmetry we choose the physical plan D_z, $y \geq 0$, limited by the contour (A_0OBCDA) and consequently D_f the strip $\psi = 0$ (A_0C) and $\psi = \frac{Q}{2}(AD)$, $\infty < \varphi < \infty$, where Q is the flow rate and in the point O we have $V = V(0)$, $p = p_0$, $\rho = \rho_0 =$ const. The Jukovski function $\omega = t + i\theta$ is considered; it is analytic with respect to $f = \varphi + i\psi$ and is a conformal mapping for $D_f \longleftrightarrow D_\omega$.

$$\omega = t + i\theta, \quad w = V^\infty e^{-\omega}, \quad t = \ln\frac{V^\infty}{V}, \quad \omega_I = 0, \quad f_\infty = 0. \tag{2}$$

Assuming that hypothesis (*Hyp*) is fulfilled, in the domain D_ζ, $\zeta = \xi + i\eta$, $\eta \geq 0$, we have following important theorems. [2]:

Theorem 1.

If there is a conformal map $f = f(\zeta)$, $f_\zeta \neq 0$, *from* D_f *to* D_ζ, *then exists too* $z = z(\zeta)$, *analytic* $z_\zeta \neq 0$ *which maps conformally* D_z *on* D_ζ. *In this case, from (1), passing* $f = f(\xi, \eta)$, $z = z(\xi, \eta)$ *on the streamline* $\psi = 0$, $\eta = 0$ *and if* $\frac{\partial \varphi}{\partial \eta}\Big|_{\eta=0} = 0$, *there are the geometrical relations which, by integration, give the equations of the obstacle (OB):*

$$x(\xi) = \int_{\xi_0}^{\xi} \frac{\cos\theta}{V}\, \varphi_\xi \, d\xi; \quad y(\xi) = \int_{\xi_0}^{\xi} \frac{\sin\theta}{V}\, \varphi_\xi \, d\xi. \tag{3}$$

Working this formulae requires the derivation of $w = w(\zeta)$, i.e. $\omega = \omega(\zeta)$.

Theorem 2.

If a conformal, function $f = f(\zeta)$, *exists, then also* $\omega = \omega(\zeta)$, $\omega_\zeta \neq 0$, *which maps conformably* D_ω to D_ζ.

2. The inverse problem

In view to the derivation of the analytic function $f = f(\zeta) = \varphi + i\psi$, the Dirichlet problem for $\psi = \psi(\xi, \eta)$ in D_ζ, $\eta \geq 0$ must be solved. For this problem, the conditions are: $\eta = 0$ on $(A_0C), \xi \in (-\infty, a)$ and $\psi = \frac{Q}{2}$ on (DA), $\xi \in (a, +\infty)$. As a result we have:

$$f(\zeta) = -\frac{Q}{2\pi}\ln(\zeta - a) + \frac{iQ}{2}; \qquad \frac{\partial\varphi}{\partial\xi}\Big|_{\eta=0} = \frac{Q}{2\pi}\,\frac{1}{\xi - a}, \quad \frac{\partial\psi}{\partial\eta}\Big|_{\eta=0} = 0, \tag{4}$$

where the parameter a will be determined below so that $V(\xi = -\infty) = V_1$;

$a = 1 + 8(r^{-\frac{1}{2\alpha}} - r^{+\frac{1}{2\alpha}})^{-1}$ where $r = \frac{V_1}{V^\infty}$ and the angles of the velocities are

$\theta(0) = \alpha\pi, \quad \theta(B) = \beta\pi, \; 0 < \beta < \alpha \leq \frac{1}{2}$.

The derivation of the analytic functions $\omega = \omega(\zeta)$ in D_ζ leads to the solving of the mixed problem knowing that on $(A_0O)\theta = 0$; $(OB)\theta = \theta(\xi)$; $(BC)t = 0$; $(AD)\theta = 0$; $(A_0(\xi = -\infty),\ O(\xi = -1),\ B(\xi = 1),\ C = D(\xi = a > 1),\ A(\xi = +\infty))$. We write the solution of the problem, [2]:

$$\omega(\zeta) = \frac{\sqrt{(1-\zeta)(a-\zeta)}}{\pi}\left[\int_{-1}^{1}\frac{\theta(s)ds}{\sqrt{(1-s)(a-s)}(s-\zeta)} + C\right] = t + i\theta$$

and then its from for $\xi \in (-1, 1)$, using Sohotski-Plemelj formulae, [1], [4]:

$$t(\theta(\xi)) = \frac{\sqrt{(1-\xi)(a-\xi)}}{\pi}\left(\int_{-1}^{1}\frac{\theta(s)}{\sqrt{(1-s)(a-s)}\,(s-\xi)}ds + C\right). \tag{5}$$

In this case of the inverse problem, if on (OB) is prescribed $V = V(\theta(\xi))$, i.e. with (2), $t = t(\theta)$, then from (5), solving the singular integral equation (in Cauchi meaning), one obtains $\theta = \theta(\xi)$, then $t = (\xi)$ and $V = V(\xi)$. Finally, from the relations (3), taking $\xi_0 = -1$, one derives the geometrical equations of the airfoil. Next, we shall consider two cases. **1-st case.** With the choice: $V = V^\infty\left(\frac{V_0}{V^\infty}\right)^{\frac{\theta-\beta\pi}{\pi(\alpha-\beta)}}$, where for $\xi = -1$, $V = V_0 << V^\infty$, it results $t = \frac{\theta - \beta\pi}{\pi(\alpha-\beta)}\ \ln\frac{V^\infty}{V_0} = m\theta + \eta$ and C is determined from (5), with $t(-1) = \ln\frac{V^\infty}{V_0}$.

In this situation, the integral equation becomes linear and is solved directly, [4], [2] or numerically. **2-nd case.** With semiinverse method, we prescribe on (OB), $\xi \in [-1, 1]$:

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$$\theta(\xi) = \pi(\beta-\alpha)\frac{\sqrt{(a-\xi)(1+\xi)}}{2(a-1)} - \alpha\pi \rightarrow V(\xi) = V^{o} e^{\frac{\pi(\alpha-\beta)}{\sqrt{2(a-1)}}\sqrt{(1-\xi)(a-\xi)}} \times \tag{6}$$

$$\times \left(\frac{\sqrt{(a-1)}\,\sqrt{a+\xi}}{\sqrt{2(a-\xi)}+\sqrt{(a+1)(1-\xi)}} \right)^{2\alpha}$$

with $V(0)=0$, the constant C=0 end with (5) one obtains $t=t(\xi)$ and $V=V(\theta)$ the velocity distribution along the airfoil. The formulae (3), normed, $X(\xi)=\frac{x(\xi)}{l}$; $Y(\xi)=\frac{y(\xi)}{l}$, where $l=\int_{-1}^{1}\frac{\varphi_{\xi}}{V(\xi)}d\xi$ is the length of the OB arc, give the parametric equations of the airfoil. [2].

The resultant of pressures acting on (OB) and the drag coefficient are:

$$P = \int_{0}^{\beta} \frac{\rho(V^{o})^2}{2}\left(1-\left(\frac{V}{V^{o}}\right)^2\right)dy; \quad C_x = \frac{P}{\frac{\rho V^{o2} l}{2}} = \frac{\int_{-1}^{1}\left(1-\left(\frac{V(\xi)}{V^{o}}\right)^2\right)\frac{\sin\theta(\xi)}{V(\xi)(a-\xi)}d\xi}{\int_{-1}^{1}\frac{d\xi}{V(\xi)(a-\xi)}}$$

All these results can be worked numerically for the design of the airfoil and for the computation of p and C_x, prescribing $V_1 < V^{o}$ (subsonic) and $0<\beta<\alpha\leq\frac{1}{2}$, [2]. For example, in the case $\alpha=\frac{1}{2}$, $\beta=\frac{1}{6}$, $r=\frac{V_1}{V^{o}}=\frac{1}{2}$ it results $C_x=0.4198$.

These inverse problem results can be subject to further research in order to archive the optimal control of the minimal drag or they can be used as a gross approximation for axially - symmetrical spatial problems in the case of compressible flow [2].

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