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The $SO_q(N)$-approach to the q-deformation of the free-particle description

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Abstract

Proofs have been given that non-commutative geometry and differential calculus can be joined together by invoking an underlying quantum-group symetry. Choosing the quantum group $SO_q(N)$, we then have to proceed by using the corresponding R-matrix solution to the parameter dependent Yang-Baxter equation. This results in a non-trivial q-deformation of the Laplacian acting on the N-dimensional non-comutative quantum Euclidian-space $R^N{}_q$. Surprisingly enough, the radial reduction of the covariant derivative implied in this manner reproduces the q-difference derivative presented long ago by Jackson. This opens the way to derive nontrivial q-deformations of the eigenvalues of the second-order Casimirs of $SO_q(N)$. The representation-dependence of q-deformed eigenvalues reffered to above is also discussed in some more detail. The free particle can then be treated in terms of q-Jackson-Bessel functions.

1. Introduction

Parameter-dependent generalisations of the usual quantum-mechanical description have attracted much interest during the last decade [1]. The deformation parameter is denoted by q, such that the usual theory gets recovered as $q \to 1$. This notation is reminiscent to the q-difference formula

$$\partial_J f(x) = \partial_q^{(x)} f(x) = \frac{f(qx) - f(x)}{(q-1)x} \tag{1.1}$$

written down long ago by Jackson [2]. Several q-hypergeometric functions have been discussed even much earlier (see §3.1 in [3]). However, the most important step characterising this decade is the synthesis between non-commutative geometry and the differential calculus, as presented in a clear manner by Wess and Zumino [4]. Such derivatives should then proceed covariantly with respect to certain quantum-group structures [5], like the one of the N-dimensional linear matrices $GL_q(N)$, or the one, say $SO_q(N)$, describing the rotations about the origin of the N-dimensional Euclidean space. These quantum-group symmetries are incorporated into $\hat{R}$ -matrices having the size $N^2 \times N^2$ and satisfying the parameter independent Yang-Baxter equation $\hat{R}_{12}\hat{R}_{23}\hat{R}_{12}=\hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$ [6]. The conventional notation for matrices acting on tensor products of vectorial spaces is used. Then the quantum-group symmetry is exhibited by a further Yang-Baxter equation like $T_1T_2\hat{R}_{12}=\hat{R}_{12}T_1T_2$ [4,6], where the linear transformations of underlying non-commutative coordinates proceed as $x^i=T^i_j x^j$. The summation over repeated covariant and contravariant indices is assumed, as usual. It should be mentioned that that the q-parameter is responsible for the non-commutative behavior of coordinates, such as given by the typical equations

$$\begin{aligned} x^1x^2&=qx^2x^1\\ x^2x^3&=qx^3x^2\\ x^1x^3-x^3x^1&=\frac{1-q}{\sqrt{q}}x^2x^2 \end{aligned} \tag{1.2}$$

characterising $SO_q(3)$, such that hereafter $q>0$. Using the q-dependent metric tensor C_{ij}, then gives the square length as [7]

$$\begin{aligned} r_q^2\equiv\bar{r}^2=C_{ij}x^j=\\ =q^{-1/2}x^1x^3+x^2x^2+q^{1/2}x^3x^1 \end{aligned} \tag{1.3}$$

for N=3, where $x_i=C_{ij}x^j$ and $[r_q^2,x^i]=0$. Starting from the basic covariant derivative formula [4]

$$\partial_l x^k=\delta_l^k+q\hat{R}_{ln}^{km}x^n\partial_m \tag{1.4}$$

opens the way to define the q-deformed Laplacian as [7-10]

$$\Delta_q=\partial^i\partial_i \tag{1.5}$$

where $\partial^i=C^{ij}\partial_j$ and $\partial_i=\partial/\partial x^i$, so that $[\Delta_q,\partial^i]=0$. Moreover, introducing algebraically the radial coordinate by virtue of the commutation relation $[r_q,x^i]=0$, nothing prevents us from defining the radial derivative by virtue of the relationship [11]

$$\partial_i=\frac{x_i}{r}\partial \tag{1.6}$$

which proceeds in a close analogy with the classical description.

Proofs have also been given that the harmonic oscillator [7-9] and the Coulomb-problem [10-12] can be solved on the N-dimensional non-commutative Euclidean space referred to above. In addition, q-deformed radial Schrödinger-equations have been written down, as shown by (34)-(36) in [11]. These equations rely on (2.33) in [8], but further explanations and/or clarifications are still in order. This conjecture motivates us to present further details and relationships concerning the q-deformation of the radial Schrödinger-equation. In this context both Hermitian and non-Hermitian q-deformed Schrödinger-Hamiltonians will be discussed. The q-deformed free particle will also be treated in some more detail. Units for which $\hbar=m_0=c=1$ are used, whereas the q-number reads

$$[[n]]_q=q^{\frac{n-1}{2}}[n]_{\sqrt{q}}=\frac{q^n-1}{q-1} \ . \tag{1.7}$$

The paper is organised as follows. The Hermitian radial momentum operator is introduced in section 2 with the help of the radial q-deformed Heisenberg-algebra. Section 3 deals with an alternative derivation of Hermitian but parameter - and system dependent Hamiltonians. The free particle is then discussed in sections 5 an 6 in terms of q-Jackson-Bessel and q-exponential functions, respectively. Section 6 contains useful details concerning the angular part of the wavefunctions and the q-deformed Casimir-eigenvalues. We conclude with a brief summary of main results and with a succinct presentation of open perspectives.

2. The q-deformed radial momentum operator

Using the concrete forms of C- and $\hat{R}$ -matrices [6,7], it can be verified that the $\tilde{\partial}$ -derivative proceeds as [11]

$$\tilde{\partial}=\tilde{\partial}_q^r=\frac{\mu}{q+1}\partial_q^{(r)} \qquad (2.1)$$

in accord with (1.1) and (1.6), where $\mu=1+q^{2-N}$. This means that $\partial_q^{(r)}=d_q/d_q r$ stands properly for the q-deformation of the usual radial derivative $\partial=d/dr$. For convenience we shall define, however, the q-deformed radial momentum-operator as

$$\tilde{p}=-i\tilde{\partial} \qquad (2.2)$$

so that

$$\tilde{p}\tilde{r}-q\tilde{r}\tilde{p}=-i\frac{\mu}{q+1} \ . \qquad (2.3)$$

Performing the Hermitian-conjugation of (2.3) and using

$$\tilde{\partial}\tilde{r}^n=\frac{\mu}{q+1}[[n]]_q\tilde{r}^{n-1}+q^n\tilde{r}^n\tilde{\partial} \qquad (2.4)$$

yields

$$\tilde{r}\tilde{p}^{+}-q\tilde{p}^{+}\tilde{r}=i\frac{\mu}{q+1} \qquad (2.5)$$

where we have assumed quite reasonably that $\tilde{r}^{+}=\tilde{r}$. Then the Hermitian conjugated radial momentum-operator is given by

$$\tilde{p}^{+}=i\tilde{\partial}^{+}=-iq^{-N}\tilde{\partial}_{q'} \qquad (2.6)$$

where now $q'=1/q$. This enables us to introduce the Hermitian q-deformed radial momentum-operator as follows

$$p_{H}=\frac{q}{\mu}(\tilde{p}+\tilde{p}^{+}) \qquad (2.7)$$

which is invariant under $q\to 1/q$. Accordingly

$$p_{H}f(\tilde{r})=-\frac{i}{\tilde{r}}\frac{f(q\tilde{r})-f(q^{-1}\tilde{r})}{q-q^{-1}} \qquad (2.8)$$

which corresponds e.g. to eq.(2.4b) in [13] and which provides a symmetrized version of the q-deformed radial derivative. Other symmetrizations have been done for the components of the vectorial momentum-operator [12,14], but this time one proceeds in conjunction with suitable star-conjugations.

We are now able to realise that the q-deformations of the three typical radial Schrödinger-equations written down before [11] exhibit inter-related Hamiltonians which are not Hermitian ones. Thus the q-deformed Hamiltonian characterising the former u-representation (see (36) in [11]) reads

$$H_{\varphi}^{(q)}=F(q)\tilde{\partial}^{2}+\frac{\lambda_{\varphi}^{(q)}}{r^{2}}+V(r) \qquad (2.9)$$

where

$$F(q)=-\frac{q}{(q+1)^{2}}(q^{-L/2}+q^{L/2})^{2} \qquad (2.10)$$

and

$$\lambda_{\varphi}^{(q)}=-\frac{\mu^2 q^2}{(q+1)^4}[[-2l-N+1]]_q[[2l+N-3]]_q\ . \qquad (2.11)$$

One has $L=l+(N-2)/2$, whereas $l=0,1,2,\ldots$, as usual. Further one obtains

$$H_{\varphi}^{(q)}-H_{\varphi}^{(q)+}=F(q)\left(\tilde{\partial}_q^{(r)2}-q^{-2N}\tilde{\partial}_{q'}^{(r)2}\right) \qquad (2.12)$$

by virtue of (2.6). Applying this difference to a monomial yields

$$\left(H_{\varphi}^{(q)}-H_{\varphi}^{(q)+}\right)r^n=G_n(q)\,r^{n-2}=$$
$$=\frac{\mu^2}{(q+1)^2}[[n]]_q[[n-1]]_q(1-q^{1-2n})r^{n-2} \qquad (2.13)$$

so that $G_n(1)=0$, as one might expect. This means that the classical limit of (2.9), i.e.

$$H_{\varphi}^{(1)}=-\partial^2+\left(L^2-\frac{1}{4}\right)\frac{1}{r^2}+V(r) \qquad (2.14)$$

is Hermitian. However, if $q\neq 1$, one has $G_n(q)=0$ if $n=0,1/2$ an 1 only. This indicates that $H_{\varphi}^{(q)}$ ceases to be Hermitian, in contradistinction to $H_{\varphi}^{(1)}$.

Such q-deformed Hamiltonians are, however, meaningful as they play the role of well defined q-parameter dependent generalisations of Hermitian classical counterparts [7,8,10]. Moreover, it has been found that they work safely in concrete applications, like the q-deformation of classical duality transformations [15]. It should also be stressed that such developments serve as a theoretical basis to the derivation of more general q-difference and/or discrete equations. In this respect we have to mention, that quantum mechanics on a lattice can be viewed as a nontrivial q-deformation [16], too. Moreover, there are reasons to consider that the flexibility of the usual quantum-mechanical description gets enhanced by accounting for appropriate non-Hermitian generalisations [17].

3. The q-deformed free particle description

After having been arrived at this stage, we are ready to analyse in some more details the q-deformed free particle description. Usual results have then to be reproduced as q→1. For this purpose, let us consider the q-deformed radial Schrödinger-equation [8,11,18]

$$H_l^{(q)} f_q^{(l)}(\tilde{r}) = (-\Delta_q^{(l,N)} + V(r)) f_q^{(l)}(\tilde{r}) = E_q f_q^{(l)}(\tilde{r}) \qquad (3.1)$$

where now

$$\Delta_q^{(l,N)} = q^{2l+N-1}\bar{\partial}^2 + \frac{\mu}{q+1}[[2l+N-1]]_q \frac{1}{\tilde{r}}\bar{\partial} \qquad (3.2)$$

which relies on (2.9) and which works in the f-representation discussed before [11]. Putting V(r)=0, we are then faced with the q-deformed eigenvalue equation

$$-\Delta_q^{(l,N)} f_q^{(\pm L)}(\tilde{r},k) = E_q f_q^{(\pm L)}(\tilde{r},k) \quad . \qquad (3.3)$$

The eigenvalue and the eigenfunctions are then given by

$$E_q = \frac{1}{4}\mu^2 k^2 \qquad (3.4)$$

where k denotes the continuous and positive momentum parameter and

$$f_q^{(\pm L)}(\tilde{r},k) = \tilde{r}^{-L} J_{\pm L}^{(1)}(k\tilde{r};q^2) \qquad (3.5)$$

respectively [18]. One has

$$J_\nu^{(1)}(x;q) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+\nu}}{[[n]]_q! \Gamma_q(n+1+\nu)} \qquad (3.6)$$

where $0<q<1$, which relies on the well known q-Jackson-Bessel functions [19-24]. We have to note, however, that (3.6) corresponds to $J_\nu^{(1)}((1-q)x;q)$, as used in the mathematical literature [24].

In addition, the q-Gamma-function quoted above fulfils the typical property (see e.g. [25])

$$\Gamma_q(n+1+\nu)=[[n+\nu]]_q\Gamma_q(n+\nu) \quad . \tag{3.7}$$

We then have to recognize that eqs.(3.3)-(3.6) express a meaningful q-deformation of the usual free particle description. Accordingly, the usual results get reproduced safely as soon as $q\to 1$.

Further clarifications are still in order. Indeed, accounting for (1.1), we have to realise that the inverse operation is given quite consistently by the q-integral [26]

$$\int_0^\infty f(\bar{r})\,d_q\bar{r}=(1-q)\sum_{j=-\infty}^{j=\infty} q^j f(q^j) \tag{3.8}$$

where $0<q<1$. Relatedly, let us consider the q-deformed sine-function

$$\sin_q x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{[[2n+1]]_q!}\frac{\pi^{1/2}}{\Gamma_{q^2}(1/2)}\left(\frac{q+1}{2}\right)^{2n+1} \quad . \tag{3.9}$$

We then have to realise that

$$\int_0^\infty \sin_q(k'\bar{r})\sin_q(k\bar{r})\,d_q\bar{r}=\frac{\pi}{2}\delta_q(k'-k) \tag{3.10}$$

stands for a nontrivial q-deformation of the usual Dirac-function

$$\int_0^\infty \sin(k'r)\sin(kr)\,dr=\frac{\pi}{2}\delta(k'-k) \quad . \tag{3.11}$$

On the other hand there is (see also (3.17) in [25])

$$\sin_q(x)=\left(\frac{1}{2}x\right)^{1/2} J_{1/2}^{(1)}(x;q^2) \tag{3.12}$$

so that the scalar product (3.10) becomes

$$(k'k)^{1/2}\langle k',1/2|k,1/2\rangle_q=\delta_q(k'-k) \tag{3.13}$$

where $|k,1/2\rangle_q$ stands for $f_q^{(1/2)}(\bar{r},k)$. Therefore, $f_q^{(1/2)}(\bar{r},k)$ exhibits definitely a well defined q-deformed version of the orthogonality condition characterising the classical solution

$f_1^{(1/2)}(r,k)$. Thus there are valuable reasons to consider that the spectrum characterising the q-deformed free particle presented above is definitely continuous, which agrees with the very form of the q-energy exhibited by (3.4). Such results are in accord with the eigenvalue-equations done recently in terms of a star symmetrized free-particle Hamiltonian (see (15)-(16) in [12]). The same remains valid with the free particle description [27] based on homogeneous spaces [28] of the Euclidean quantum group $E_q(2)$.

Nevertheless, a discrete free-particle spectrum has been derived by looking from the very beginning for normalizable power-series expansions without negative powers in the radial coordinate [29]. The point is that the discrete energy derived in this manner behaves like $1/(1-q^2)^2$, so that it does not express, as a matter of fact, an actual q-deformation of the classical result. Such results should then be viewed as exclusive manifestations implied by the quantum-group description, so that they deserve further attention.

4. The q-deformed centrifugal barriers

Now we would like to use this opportunity to clarify several aspects concerning the q-deformed centrifugal barriers as well as the angular-dependence of total wavefunctions. First we have to recall that the radial unrenormalized wavefunctions characterising (34)-(36) in [11] are inter-related as

$$\psi_q^{(l)}(\tilde{r})=\tilde{r}^{\,l} f_q^{(l)}(\tilde{r})=\tilde{r}^{\,l-a}\varphi_q^{(l)}(\tilde{r}) \qquad (4.1)$$

where

$$q^{a}=\frac{q^{-2L}+1}{q+1} \qquad (4.2)$$

and where $\psi_q^{(l)}(\tilde{r})$ and $\varphi_q^{(l)}(\tilde{r})$ stand for the former g(r) and f(r), respectively. Starting from $f_q^{(l)}(\tilde{r})$, we can then say that the total q-deformed wavefunction factorizes as

$$\Psi_q^{(l)}(\vec{x})=S_l^{(l)}(x^i)\,f_q^{(l)}(\bar{r}) \tag{4.3}$$

where $S_l^{(l)}(x^i)$ is a suitably symmetrized sum of monomials of the same degrees in the non-commutative x^i-coordinates (see (2.23) in [8]). On the other hand (4.3) can be rewritten equivalently as

$$\Psi_q^{(l)}(\vec{x})=Y_q^{(l)}(\xi^i)\,\psi_q^{(l)}(\bar{r}) \tag{4.4}$$

where $\xi^i=x^i/\bar{r}$ and where

$$Y_q^{(l)}(\xi^i)=\frac{1}{\bar{r}^l}S_l^{(l)}(x^i) \tag{4.5}$$

is the q-deformed counterpart of N-dimensional spherical harmonics, i.e. of Gegenbauer-polynomials. Furthermore one has

$$\begin{aligned}\Delta_q\left(S_l^{(l)}(x^i)\,f_q^{(l)}(\bar{r})\right)&=S_l^{(l)}(x^i)\,\Delta_q^{(l,N)}f_q^{(l)}(\bar{r})=\\&=S_l^{(l)}(x^i)\left(q^{2l}\Delta_q+\frac{\mu}{q+1}[[2l]]_q C_{ij}x^i\partial^j\right)f_q^{(l)}(\bar{r})\end{aligned} \tag{4.6}$$

which leads to the q-deformed eigenvalue-equation

$$-\Delta_q Y_q^{(l)}(\xi^i)=\lambda_\psi^{(q)}Y_q^{(l)}(\xi^i) \tag{4.7}$$

via $f_q^{(l)}(\bar{r})=\bar{r}^{-l}$. We have to remark that

$$\lambda_\psi^{(q)}=\frac{\mu^2}{(q+1)^2}q^{-l}[[l]]_q[[l+N-2]]_q \tag{4.8}$$

reproduces precisely the amplitude of the q-deformed centrifugal barrier in (35) in [11], as one might expect. Of course, (4.8) stands for the q-deformed eigenvalue of the square angular momentum in the ψ-representation. Concrete form of such q-deformed spherical harmonics have already been written down for l=0,1 an 2 [10,12]. Moreover, certain q-generalisations of Gegenbauer polynomials have also been discussed [30]. It should be mentioned that in order to treat (4.6) we have to resort basically to (1.4), but further equations like

$$\bar{\partial}^i x^j = C^{ij} + q(\hat{R}^{-1})^{ij}_{ml} x^m \bar{\partial}^l \tag{4.9}$$

and

$$\partial_i x_j = C_{ji} + q(\hat{R}^{-1})^{kl}_{ji} x_l \partial_k \tag{4.10}$$

are useful.

Proceeding similarly and putting $f_q^{(l)}(\bar{r}) = \bar{r}^a$ yields the q-deformed eigenvalue (2.11). It should also be mentioned that (2.11) and (4.8) correspond to the q-deformed Casimir-eigenvalues presented before for N=3 [12,31] and for arbitrary N [29], respectively. Thus we succeeded to clarify the representation background of the two different q-deformed Casimir-eigenvalues discussed separately before.

5. Conclusions

In this paper we have discussed further aspects concerning the q-deformation of the radial Schrödinger-equation. So the Hermitian radial momentum operator (2.6) has been derived in a quite transparent manner with the help of the radial q-deformed Heisenberg-algebra (2.3). Closed free-particle solutions to the q-deformed radial Schrödinger-equation (3.1) have been written down. We have also succeeded in clarifying several details concerning the angular part of the q-deformed wavefunctions as well as the influence of the representation on the q-deformed Casimir-eigenvalues. In general, q-deformed radial Schrödinger-equations mentioned above are not easily solvable, but useful approximations can be derived by applying q-deformed 1/N-formulae [11,15,32]. Besides (2.3), other generalisations of the Heisenberg-algebra have also been discussed [33].

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