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Runge-Kutta method for second order ordinary differential equation.

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In this paper are presented explicit fourth order Runge-Kutta method for $y'' = f(x,y,y')$. We have derived thirtyfive condition equations for twentyfour parameters. We have solution for this condition equations where is the numerical method for second order ordinary differential equation with rational coefficients.

We shall consider initial value problem for the second order ordinary differential equation:

$$y'' = f(x,y,y'), \qquad y(x_0) = y_0, \qquad y'(x_0) = y'_0 \tag{1}$$

and consider a general three-stage explicit Runge - Kutta method defined by:

$$\begin{aligned} y_{k+1} &= y_k + hy'_k + h^2(a_1K_1 + a_2K_2 + a_3K_3 + a_4K_4) + O(h^5) \\ y'_{k+1} &= y'_k + h(b_1K_1 + b_2K_2 + b_3K_3 + b_4K_4) + O(h^5) \end{aligned} \tag{2}$$

where

$$\begin{aligned} K_1 &= f[x_k + \alpha_1 h, y_k + \alpha_1 hy'_k, y'_k], \\ K_2 &= f[x_k + \alpha_2 h, y_k + \alpha_2 hy'_k + h^2\beta_{21}K_1, y'_k + h\gamma_{21}K_1], \\ K_3 &= f[x_k + \alpha_3 h, y_k + \alpha_3 hy'_k + h^2(\beta_{31}K_1 + \beta_{32}K_2), y'_k + h(\gamma_{31}K_1 + \gamma_{32}K_2)], \\ K_4 &= f[x_k + \alpha_4 h, y_k + \alpha_4 hy'_k + h^2(\beta_{41}K_1 + \beta_{42}K_2 + \beta_{43}K_3), y'_k + h(\gamma_{41}K_1 + \gamma_{42}K_2 + \gamma_{43}K_3)] \end{aligned} \tag{3}$$

We develop of $y_{k+1}, y'_{k+1}, K_1, K_2, K_3$ and K_4 into the Taylor series but we use following simplifications:

$$Df = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y}, \qquad D^2f = \frac{\partial^2 f}{\partial x^2} + 2y'\frac{\partial^2 f}{\partial x\partial y} + y'^2\frac{\partial^2 f}{\partial y^2}, \qquad Df_1 = \frac{\partial^2 f}{\partial x\partial y'} + y'\frac{\partial^2 f}{\partial y\partial y'},$$

$$D^3f = \frac{\partial^3 f}{\partial x^3} + 3y'\frac{\partial^3 f}{\partial x^2\partial y} + 3y'^2\frac{\partial^3 f}{\partial x\partial y^2} + y'^3\frac{\partial^3 f}{\partial y^3}, \qquad Df_2 = \frac{\partial^2 f}{\partial x\partial y} + y'\frac{\partial^2 f}{\partial y^2},$$

$$D^2f_1 = \frac{\partial^3 f}{\partial x^2\partial y'} + 2y'\frac{\partial^3 f}{\partial x\partial y\partial y'} + y'^2\frac{\partial^3 f}{\partial y^2\partial y'}, \qquad Df_3 = \frac{\partial^3 f}{\partial x\partial y'^2} + y'\frac{\partial^3 f}{\partial y\partial y'^2},$$

$$y_{k+1} = y(x_k + h) = y(x_k) + hy' + \frac{h^2}{2!} f + \frac{h^3}{3!}\left[Df + f\frac{\partial f}{\partial y'} \right] + \frac{h^4}{4!}[D^2 f + 2f.Df_1 + Df.\frac{\partial f}{\partial y'} +$$

$$+ f\frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y'}\right)^2 + f^2\frac{\partial^2 f}{\partial y'^2}] + O(h^5),$$

$$y'_{k+1} = y'(x_k + h) = y'(x_k) + hf + \frac{h^2}{2!}\left[Df + f\frac{\partial f}{\partial y'} \right] + \frac{h^3}{3!}\ [\ D^2 f + 2f.Df_1 + Df.\frac{\partial f}{\partial y'} + f\frac{\partial f}{\partial y} +$$

$$+ f\left(\frac{\partial f}{\partial y'}\right)^2 + f^2\frac{\partial^2 f}{\partial y'^2}\ ] + \frac{h^4}{4!}\ [\ D^3 f + 3f.Df_2 + 3f.D^2 f_1 + 3Df.Df_1 + 5f.Df.Df_1 + D^2 f\frac{\partial f}{\partial y'} +$$

$$+ 3f.Df\frac{\partial^2 f}{\partial y'^2} + 3y'.Df\frac{\partial^2 f}{\partial y \partial y'} + 3f^2.Df_3 + Df.\frac{\partial f}{\partial y} + Df\left(\frac{\partial f}{\partial y'}\right)^2 + 2f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y'} + 3f^2\frac{\partial^2 f}{\partial y \partial y'} +$$

$$+ f\left(\frac{\partial f}{\partial y'}\right)^3 + 4f^2\frac{\partial f}{\partial y'}\frac{\partial^2 f}{\partial y'^2} + f^3\frac{\partial^3 f}{\partial y'^3}\ ] + O(h^5),$$

$$K_1 = f + h.\alpha_1.Df + \frac{h^2}{2!}.\alpha_1^{\ 2}.D^2 f + \frac{h^3}{3!}.\alpha_1^{\ 3}.D^3 f + O(h^4),$$

$$K_2 = f + h\left[\alpha_2 Df + \gamma_{21} f\frac{\partial f}{\partial y'} \right] + \frac{h^2}{2!}[\ \alpha_2^{\ 2} D^2 f + 2\alpha_2\gamma_{21} f.Df_1 + 2\alpha_1\gamma_{21} Df\frac{\partial f}{\partial y'} + 2\beta_{21} f\frac{\partial f}{\partial y} +$$

$$+ \gamma_{21}^{\ 2} f^2\frac{\partial^2 f}{\partial y'^2}\ ] + \frac{h^3}{3!}\ [\ \alpha_2^{\ 3} D^3 f + 3\alpha_2^{\ 2}\gamma_{21} f.D^2 f_1 + 3\alpha_2\gamma_{21}^{\ 2} f^2.Df_3 + \gamma_{21}^{\ 3} f^3\frac{\partial^3 f}{\partial y'^3} +$$

$$+ 6\alpha_1\beta_{21} Df\frac{\partial f}{\partial y} + 3\alpha_1^{\ 2}\gamma_{21} D^2 f\frac{\partial f}{\partial y'} + 6\alpha_2\beta_{21} f.Df_2 + 6\alpha_1.\alpha_2\gamma_{21} Df.Df_1 + 6\alpha_1\gamma_{21}^{\ 2} f.Df.\frac{\partial^2 f}{\partial y'^2} +$$

$$+ 6\beta_{21}\gamma_{21} f^2\frac{\partial^2 f}{\partial y \partial y'}\ ] + O(h^4),$$

$$K_3 = f + h\left[\alpha_3.Df + (\gamma_{31} + \gamma_{32})\frac{\partial f}{\partial y'} \right] + \frac{h^2}{2!}\ [\ \alpha_3^{\ 2} D^2 f + 2(\alpha_1\gamma_{31} + \alpha_2\gamma_{32}).Df.\frac{\partial f}{\partial y'} +$$

$$+ 2(\beta_{31} + \beta_{32}) f\frac{\partial f}{\partial y} + 2\alpha_3(\gamma_{31} + \gamma_{32}) f.Df_1 + 2\gamma_{21}\gamma_{32} f\left(\frac{\partial f}{\partial y'}\right)^2 + (\gamma_{31} + \gamma_{32})^2 f^2\frac{\partial^2 f}{\partial y'^2}\ ] +$$

$$+ \frac{h^3}{3!}\ [\ \alpha_3^{\ 3} D^3 f + 3\alpha_3^{\ 2}(\gamma_{31} + \gamma_{32}) f.D^2 f_1 + 3\alpha_3(\gamma_{31} + \gamma_{32})^2 f^2 Df_3 + (\gamma_{31} + \gamma_{32})^3 f^3\frac{\partial^3 f}{\partial y'^3} +$$

$$+ 6(\alpha_1\beta_{31} + \alpha_2\beta_{32}) Df\frac{\partial f}{\partial y} + 3\left(\alpha_1^{\ 2}\gamma_{31} + \alpha_2^{\ 2}\gamma_{32}\right) D^2 f\frac{\partial f}{\partial y'} + 6\alpha_3(\beta_{31} + \beta_{32}) f.Df_2 +$$

$$+6\alpha_3(\alpha_1\gamma_{31}+\alpha_2\gamma_{32})Df.Df_1+6(\alpha_1\gamma_{31}+\alpha_2\gamma_{32})(\gamma_{31}+\gamma_{32})f.Df.\frac{\partial^2 f}{\partial y'^2}+6\alpha_1\gamma_{21}\gamma_{32}Df\left(\frac{\partial f}{\partial y'}\right)^2+$$

$$+6(\beta_{31}+\beta_{32})(\gamma_{31}+\gamma_{32})f^2\frac{\partial^2 f}{\partial y\partial y'}+6(\alpha_2+\alpha_3)\gamma_{21}\gamma_{32}f.Df_1\frac{\partial f}{\partial y'}+6(\beta_{21}\gamma_{32}+\beta_{32}\gamma_{21})f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y'}+$$

$$+3\gamma_{21}\gamma_{32}(\gamma_{21}+\gamma_{31}+\gamma_{32})f^2\frac{\partial f}{\partial y'}\frac{\partial^2 f}{\partial y'^2}\ ]+O(h^4),$$

$$K_4=f+h\left[\alpha_4 Df+(\gamma_{41}+\gamma_{42}+\gamma_{43})f\frac{\partial f}{\partial y'}\right]+\frac{h^2}{2!}[\alpha_4^2D^2f+2(\alpha_1\gamma_{41}+\alpha_2\gamma_{42}+\alpha_3\gamma_{43})Df\frac{\partial f}{\partial y'}+$$

$$+2(\beta_{41}+\beta_{42}+\beta_{43})f\frac{\partial f}{\partial y}+2\alpha_4(\gamma_{41}+\gamma_{42}+\gamma_{43})f.Df_1+2(\gamma_{21}\gamma_{42}+(\gamma_{31}+\gamma_{32})\gamma_{43})f\left(\frac{\partial f}{\partial y'}\right)^2+$$

$$+(\gamma_{41}+\gamma_{42}+\gamma_{43})^2f^2\frac{\partial^2 f}{\partial y'^2}\ ]+\frac{h^3}{3!}[\ \alpha_4^3D^3f+3\alpha_4^2(\gamma_{41}+\gamma_{42}+\gamma_{43})f.D^2f_1+$$

$$+3\alpha_4(\gamma_{41}+\gamma_{42}+\gamma_{43})^2f^2.Df_3+(\gamma_{41}+\gamma_{42}+\gamma_{43})^3f^3\frac{\partial^3 f}{\partial y'^3}+6(\alpha_1\beta_{41}+\alpha_2\beta_{42}+\alpha_3\beta_{43})Df\frac{\partial f}{\partial y}+$$

$$+3(\alpha_1^2.\gamma_{41}+\alpha_2^2.\gamma_{42}+\alpha_3^2.\gamma_{43})D^2f\frac{\partial f}{\partial y'}+6\alpha_4(\alpha_1.\gamma_{41}+\alpha_2.\gamma_{42}+\alpha_3.\gamma_{43})Df.Df_1+$$

$$+6\alpha_4(\beta_{41}+\beta_{42}+\beta_{43})f.Df_2+6(\alpha_1\gamma_{41}+\alpha_2\gamma_{42}+\alpha_3\gamma_{43})(\gamma_{41}+\gamma_{42}+\gamma_{43})f.Df\frac{\partial^2 f}{\partial y'^2}+$$

$$+6(\beta_{41}+\beta_{42}+\beta_{43})(\gamma_{41}+\gamma_{42}+\gamma_{43})f^2\frac{\partial^2 f}{\partial y\partial y'}+6\alpha_4(\gamma_{21}\gamma_{42}+(\gamma_{31}+\gamma_{32})\gamma_{43})f.Df_1\frac{\partial f}{\partial y'}+$$

$$+6(\beta_{21}\gamma_{42}+(\beta_{31}+\beta_{32})\gamma_{43})f\frac{\partial f}{\partial y}\frac{\partial f}{\partial y'}+6(\alpha_1\gamma_{21}\gamma_{42}+(\alpha_1\gamma_{31}+\alpha_2\gamma_{32})\gamma_{43})Df.\left(\frac{\partial f}{\partial y'}\right)^2+$$

$$+3(\gamma_{21}^2\gamma_{42}+(\gamma_{31}+\gamma_{32})^2\gamma_{43})f^2.\frac{\partial f}{\partial y'}\frac{\partial^2 f}{\partial y'^2}+6(\gamma_{21}\beta_{42}+(\gamma_{31}+\gamma_{32})\beta_{43})f.\frac{\partial f}{\partial y}\frac{\partial f}{\partial y'}+$$

$$+6(\gamma_{21}\gamma_{42}+(\gamma_{31}+\gamma_{32})\gamma_{43})(\gamma_{41}+\gamma_{42}+\gamma_{43})f^2\frac{\partial f}{\partial y'}\frac{\partial^2 f}{\partial y'^2}+6\gamma_{21}.\gamma_{32}.\gamma_{43}.f.\left(\frac{\partial f}{\partial y'}\right)^3+$$

$$+6(\alpha_2\gamma_{21}\gamma_{42}+\alpha_3(\gamma_{31}+\gamma_{32})\gamma_{43})f.Df_1.\frac{\partial f}{\partial y'}\ ]+O(h^4).$$

By inserting the last Taylor series into the formulae (2) we obtain series that we compare in the same variables articles on the both side equations. We define indefinite coefficients so that the first 4-th terms of both series coincide. In this way we get the condition equations of the 4-th order and the 4-th stage :

$$
\begin{aligned}
&a_1 + a_2 + a_3 + a_4 = \frac{1}{2},\\
&a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = \frac{1}{6},\\
&a_1\alpha_1^2 + a_2\alpha_2^2 + a_3\alpha_3^2 + a_4\alpha_4^2 = \frac{1}{12},\\
&a_2\beta_{21} + a_3(\beta_{31} + \beta_{32}) + a_4(\beta_{41} + \beta_{42} + \beta_{43}) = \frac{1}{24},\\
&a_2\gamma_{21} + a_3(\gamma_{31} + \gamma_{32}) + a_4(\gamma_{41} + \gamma_{42} + \gamma_{43}) = \frac{1}{6},\\
&a_2\alpha_2\gamma_{21} + a_3\alpha_3(\gamma_{31} + \gamma_{32}) + a_4\alpha_4(\gamma_{41} + \gamma_{42} + \gamma_{43}) = \frac{1}{12},\\
&a_2\alpha_1\gamma_{21} + a_3(\alpha_1\gamma_{31} + \alpha_2\gamma_{32}) + a_4(\alpha_1\gamma_{41} + \alpha_2\gamma_{42} + \alpha_3\gamma_{43}) = \frac{1}{24},\\
&a_2\gamma_{21}^2 + a_3(\gamma_{31} + \gamma_{32})^2 + a_4(\gamma_{41} + \gamma_{42} + \gamma_{43})^2 = \frac{1}{12},\\
&a_3\gamma_{21}\gamma_{32} + a_4(\gamma_{21}\gamma_{42} + (\gamma_{31} + \gamma_{32})\gamma_{43}) = \frac{1}{24},\\
&b_1 + b_2 + b_3 + b_4 = 1,\\
&b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 + b_4\alpha_4 = \frac{1}{2},\\
&b_1\alpha_1^2 + b_2\alpha_2^2 + b_3\alpha_3^2 + b_4\alpha_4^2 = \frac{1}{3},\\
&b_1\alpha_1^3 + b_2\alpha_2^3 + b_3\alpha_3^3 + b_4\alpha_4^3 = \frac{1}{4},\\
&b_2\gamma_{21} + b_3(\gamma_{31} + \gamma_{32}) + b_4(\gamma_{41} + \gamma_{42} + \gamma_{43}) = \frac{1}{2},\\
&b_2\alpha_2\gamma_{21} + b_3\alpha_3(\gamma_{31} + \gamma_{32}) + b_4\alpha_4(\gamma_{41} + \gamma_{42} + \gamma_{43}) = \frac{1}{3},\\
&b_2\alpha_2^2\gamma_{21} + b_3\alpha_3^2(\gamma_{31} + \gamma_{32}) + b_4\alpha_4^2(\gamma_{41} + \gamma_{42} + \gamma_{43}) = \frac{1}{4},\\
&b_2\beta_{21} + b_3(\beta_{31} + \beta_{32}) + b_4(\beta_{41} + \beta_{42} + \beta_{43}) = \frac{1}{6},\\
&b_2\alpha_2\beta_{21} + b_3\alpha_3(\beta_{31} + \beta_{32}) + b_4\alpha_4(\beta_{41} + \beta_{42} + \beta_{43}) = \frac{1}{8},
\end{aligned}
\tag{4}
$$

$$b_2\alpha_1\beta_{21}+b_3(\alpha_1\beta_{31}+\alpha_2\beta_{32})+b_4(\alpha_1\beta_{41}+\alpha_2\beta_{42}+\alpha_3\beta_{43})=\frac{1}{24},$$

$$b_2\beta_{21}\gamma_{21}+b_3(\beta_{31}+\beta_{32})(\gamma_{31}+\gamma_{32})+b_4(\beta_{41}+\beta_{42}+\beta_{43})(\gamma_{41}+\gamma_{42}+\gamma_{43})=\frac{1}{8},$$

$$b_3(\beta_{21}\gamma_{32}+\beta_{32}\gamma_{21})+b_4(\beta_{42}\gamma_{21}+\beta_{43}(\gamma_{31}+\gamma_{32})+\beta_{21}\gamma_{42}+(\beta_{31}+\beta_{32})\gamma_{43})=\frac{1}{12},$$

$$b_2\gamma_{21}^2+b_3(\gamma_{31}+\gamma_{32})^2+b_4(\gamma_{41}+\gamma_{42}+\gamma_{43})^2=\frac{1}{3},$$

$$b_2\alpha_2\gamma_{21}^2+b_3\alpha_3(\gamma_{31}+\gamma_{32})^2+b_4\alpha_4(\gamma_{41}+\gamma_{42}+\gamma_{43})^2=\frac{1}{4},$$

$$b_2\gamma_{21}^3+b_3(\gamma_{31}+\gamma_{32})^3+b_4(\gamma_{41}+\gamma_{42}+\gamma_{43})^3=\frac{1}{4},$$

$$b_2\alpha_1\gamma_{21}+b_3(\alpha_1\gamma_{31}+\alpha_2\gamma_{32})+b_4(\alpha_1\gamma_{41}+\alpha_2\gamma_{42}+\alpha_3\gamma_{43})=\frac{1}{6},$$

$$b_2\alpha_1^2\gamma_{21}+b_3(\alpha_1^2\gamma_{31}+\alpha_2^2\gamma_{32})+b_4(\alpha_1^2\gamma_{41}+\alpha_2^2\gamma_{42}+\alpha_3^2\gamma_{43})=\frac{1}{12},$$

$$b_2\alpha_1\alpha_2\gamma_{21}+b_3\alpha_3(\alpha_1\gamma_{31}+\alpha_2\gamma_{32})+b_4\alpha_4(\alpha_1\gamma_{41}+\alpha_2\gamma_{42}+\alpha_3\gamma_{43})=\frac{1}{8},$$

$$b_2\alpha_1\gamma_{21}^2+b_3(\gamma_{31}+\gamma_{32})(\alpha_1\gamma_{31}+\alpha_2\gamma_{32})+b_4(\gamma_{41}+\gamma_{42}+\gamma_{43})(\alpha_1\gamma_{41}+\alpha_2\gamma_{42}+\alpha_3\gamma_{43})=\frac{1}{8},$$

$$b_3\gamma_{21}\gamma_{32}+b_4(\gamma_{21}\gamma_{42}+(\gamma_{31}+\gamma_{32})\gamma_{43})=\frac{1}{6}$$

$$b_3\alpha_1\gamma_{21}\gamma_{32}+b_4(\alpha_1\gamma_{21}\gamma_{42}+(\alpha_1\gamma_{31}+\alpha_2\gamma_{32})\gamma_{43})=\frac{1}{24},$$

$$b_3\alpha_2\gamma_{21}\gamma_{32}+b_4(\alpha_2\gamma_{21}\gamma_{42}+\alpha_3(\gamma_{31}+\gamma_{32})\gamma_{43})=\frac{1}{12},$$

$$b_3\alpha_3\gamma_{21}\gamma_{32}+b_4(\alpha_3\gamma_{21}\gamma_{42}+\alpha_4(\gamma_{31}+\gamma_{32})\gamma_{43})=\frac{1}{8},$$

$$b_3\gamma_{21}^2\gamma_{32}+b_4(\gamma_{21}^2\gamma_{42}+(\gamma_{31}+\gamma_{32})^2\gamma_{43})=\frac{1}{12},$$

$$b_3\gamma_{21}\gamma_{32}(\gamma_{31}+\gamma_{32})+b_4(\gamma_{21}\gamma_{42}+(\gamma_{31}+\gamma_{32})\gamma_{43})(\gamma_{41}+\gamma_{42}+\gamma_{43})=\frac{1}{8},$$

$$b_4\gamma_{21}\gamma_{32}\gamma_{43}=\frac{1}{24}.$$

We can see that we have thirtytwo equations of conditions necessary for order four we involving thetwentyfour indefinite coefficients.When we generalize results from the paper [2] is evidence that must be valid:

$\alpha_2 = \gamma_{21}, \alpha_3 = \gamma_{31} + \gamma_{32}, \alpha_4 = \gamma_{41} + \gamma_{42} + \gamma_{43}$ and when $\alpha_1 = 0$ than $b_1 \neq 0$.In order to derive the Runge - Kutta formulas we need to find the solution of the above mentioned condition equations (4).We seek the solution in the form of rational numbers ,because then the residues of the eguations after introducing the values are zeros and the solution is exact up to the term h^4.Next we have one solution condition equations (4):

$$a_1 = \frac{1}{24}, a_2 = \frac{8}{24}, a_3 = \frac{2}{24}, a_4 = \frac{1}{24}, \qquad \alpha_1 = 0, \alpha_2 = \frac{1}{4}, \alpha_3 = \frac{1}{2}, \alpha_4 = 1,$$

$$b_1 = \frac{1}{6}, b_2 = 0, b_3 = \frac{4}{6}, b_4 = \frac{1}{6},$$

$$\beta_{21} = \frac{1}{32}, \beta_{31} = \frac{1}{16}, \beta_{32} = \frac{1}{16}, \beta_{41} = \frac{1}{4}, \beta_{42} = -\frac{1}{4}, \beta_{43} = \frac{2}{4}, \tag{5}$$

$$\gamma_{21} = \frac{1}{4}, \gamma_{31} = 0, \gamma_{32} = \frac{1}{2}, \gamma_{41} = 1, \gamma_{42} = -2, \gamma_{43} = 2.$$

$$y_{k+1} = y_k + h.y'_k + \frac{h^2}{24}(K_1 + 8K_2 + 2K_3 + K_4),$$

$$y'_{k+1} = y'_k + \frac{h}{6}(K_1 + 4K_3 + K_4), \text{where} \tag{6}$$

$$K_1 = f(x_k; y_k; y'_k),$$

$$K_2 = f\left(x_k + \frac{h}{4}; y_k + \frac{h}{4}y'_k + \frac{h^2}{32}K_1; y'_k + \frac{h}{4}K_1\right),$$

$$K_3 = f\left(x_k + \frac{h}{2}; y_k + \frac{h}{2}y'_k + \frac{h^2}{16}(K_1 + K_2); y'_k + \frac{h}{2}K_2\right),$$

$$K_4 = f\left(x_k + h; y_k + hy'_k + \frac{h^2}{4}(K_1 - K_2 + 2K_3); y'_k + h(K_1 - 2K_2 + 2K_3)\right).$$

This procedure yields the Runge - Kutta formulae (5) and (6) of the 4-th order and 4-th stage for the solution of the ordinary differential equation of the second order,which is easily programmable.

References

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