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ASYMPTOTIC PROPERTIES OF OSCILLATORY SOLUTIONS OF N-TH ORDER NONLINEAR DIFFERENTIAL EQUATION WITH DEVIATING ARGUMENT

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Abstract

This paper deals with the conditions under which all oscillatory solutions of the n-th order nonlinear differential equation with quasiderivatives and deviating argument of a form

$$(r_{n-1}(t)(\dots r_2(t)(r_1(t)y'(t))'\dots)')' + a(t)f(y(g(t))) = b(t)$$

tend to zero for $t \to \infty$.
Keywords: n-th order, nonlinear, differential equation, quasiderivatives, deviating argument, oscillatory solutions, asymptotic properties.

1. Introduction

In this paper we shall investigate the behavior of the oscillatory solutions of the n-th order nonlinear differential equation with quasiderivatives and deviating argument of a form

$$L_n y(t) + a(t)f(y(g(t))) = b(t) \tag{1}$$

on some unbounded $\langle t_0, \infty)$, where the so called quasiderivatives are defined by the following relations

$$L_0 y(t) = y(t)\,,\quad L_i y(t) = r_i(t)[L_{i-1}y(t)]' \text{ for } i = 1, 2, \dots, n-1\,, \tag{2}$$
$$L_n y(t) = [L_{n-1}y(t)]'\,.$$

We shall consider only so called regular solutions of this equation, i.e. solutions which are defined on some neighbourhood of infinity and which are not eventually trivial. Such a solution is said to be oscillatory if it has arbitrarily large zeroes, otherwise it is said to be nonoscillatory.

The aim of this paper is to give some sufficient conditions under which each oscillatory solution of the equation (1) tends to zero for $t \to \infty$.

Let $D(L_n, T)$ be a set of all functions y such that $L_i y(t)$ is defined on $\langle T, \infty)$ for $i = 0, 1, \dots, n$. Then the domain of the differential operator L_n is a set

$$D(L_n) = \bigcup_{T \geq t_0} D(L_n, T)\,.$$

Through the whole paper we shall assume the following conditions to be fulfilled.

$$\left.\begin{array}{l}\text{(a) } n \geq 2 \text{ is integer,} \\ \text{(b) } r_1, r_2, \ldots, r_{n-1}, a, b, g \in C\langle t_0, \infty), \; f \in C(-\infty, +\infty), \\ \text{(c) } r_i(t) > 0 \quad \text{for} \quad t \geq t_0, \; i = 1, 2, \ldots, n-1, \\ \text{(d) } \lim_{t\to\infty} g(t) = \infty, \\ \text{(e) } f(y) \text{ is nondecreasing and } yf(y) > 0 \text{ for } y \neq 0. \end{array}\right\} \tag{3}$$

Let us define the following functions and operators.

$$\left.\begin{array}{l}\text{(a) } R_0(t) = 1, \\ \text{(b) } R_i(t) = \int\limits_t^{\infty} \frac{R_{i-1}(s)}{r_i(s)} ds \text{ if } \int\limits_{t_0}^{\infty} \frac{R_{i-1}(s)}{r_i(s)} ds < \infty, \\ \text{(c) } R_i(t) = \int\limits_{t_0}^{t} \frac{R_{i-1}(s)}{r_i(s)} ds \text{ if } \int\limits_{t_0}^{\infty} \frac{R_{i-1}(s)}{r_i(s)} ds = \infty \end{array}\right\} \tag{4}$$

for $t \geq t_0$ and $i = 1, 2, \ldots, n-1$. Let $y \in D(L_n, T)$ for some $T \geq t_0$, then we shall define

$$U_i(y, T, t) = \int\limits_T^t R_i(s)[L_i y(s)]' ds \text{ for } t \geq T, \; i = 0, 1, \ldots, n-1. \tag{5}$$

2. Auxiliary assertions

LEMMA 1. *Let $y(t) \in D(L_n, T)$ for some $T \geq t_0$, then*

a) if $\lim_{t\to\infty} U_{n-1}(y, T, t)$ exists (proper or improper), then $\lim_{t\to\infty} y(t)$ exists (proper or improper), too,

b) if $|\lim_{t\to\infty} U_{n-1}(y, T, t)| = \infty$, then $|\lim_{t\to\infty} y(t)| = \infty$, too.

PROOF. The proof can be found in [2].

LEMMA 2. *Let*

$$\int\limits_{t_0}^{\infty} \frac{R_{i-1}(t)}{r_i(t)} < \infty \text{ for } i = 1, 2, \ldots, n-1. \tag{6}$$

Let $y \in D(L_n, T)$ for some $T > t_0$, $S > T$ and $K > 0$. Then

a) if $U_{n-1}(y,T,t) \le K$ for $T \le t \le S$,

then $y(t) \le K + \sum_{i=0}^{n-1} |R_i(T)L_i y(T)|$ for $T \le t \le S$,

b) if $U_{n-1}(y,T,t) \ge -K$ for $T \le t \le S$,

then $y(t) \ge -K - \sum_{i=0}^{n-1} |R_i(T)L_i y(T)|$ for $T \le t \le S$.

PROOF. We shall prove the first part of the lemma, the proof of the second one is analogous. Let $i \in \{1, 2, \dots, n-1\}$, then by (5) we have

$$U_i(y,T,t) = \int_T^t R_i(s)[L_i y(s)]' ds = R_i(t)L_i y(t) - \int_T^t [R_i(s)]' L_i y(s) ds - R_i(T)L_i y(T)$$

and by (6), (2), (4)(b) and (5) we have

$$U_i(y,T,t) = R_i(t)r_i(t)[L_{i-1}y(t)]' + \int_T^t \frac{R_{i-1}(s)}{r_i(s)} r_i(s)[L_{i-1}y(s)]' ds - R_i(T)L_i y(T) =$$

$$= R_i(t)\frac{r_i(t)}{R_{i-1}(t)} R_{i-1}(t)[L_{i-1}y(t)]' + \int_T^t R_{i-1}(s)[L_{i-1}y(s)]' ds - R_i(T)L_i y(T) =$$

$$= -\frac{R_i(t)}{[R_i(t)]'}[U_{i-1}(y,T,t)]' + U_{i-1}(y,T,t) - R_i(T)L_i y(T) \text{ for } t \ge T.$$

Hence, the function $u(t) = U_{i-1}(y,T,t)$ is the solution of the 1^{st} order linear differential equation

$$u'(t) - \frac{[R_i(t)]'}{R_i(t)} u(t) + \frac{[R_i(t)]'}{R_i(t)}[U_i(y,T,t) + R_i(T)L_i y(T)] = 0$$

on $< T, \infty)$ satisfiing the condition $u(T) = 0$. Directly solving this equation we obtain the following recurent relation

$$U_{i-1}(y,T,t) = R_i(t) \int_T^t \frac{-[R_i(s)]'}{[R_i(s)]^2}[U_i(y,T,s) + R_i(T)L_i y(T)] ds \tag{7}$$

for $t \ge T$ and $i = n-1, n-2, \dots, 2, 1$.
Let $U_i(y,T,t) \le K_i$ for $T \le t \le S$ and some $K_i > 0$, then by (7) we have

$$U_{i-1}(y,T,t) \le [K_i + |R_i(T)L_i y(T)|] R_i(t) \int_T^t \frac{-[R_i(s)]'}{[R_i(s)]^2} ds =$$

$$= [K_i + |R_i(T)L_i y(T)|] R_i(t) \left[\frac{1}{R_i(t)} - \frac{1}{R_i(T)}\right] \le K_i + |R_i(T)L_i y(T)|$$

for $T \le t \le S$. Using this inequality for $i = n-1, n-2, \dots, 2, 1$ with $K_{n-1} = K$ we get

$$U_0(y,T,t) \le K + \sum_{i=1}^{n-1} |R_i(T)L_i y(T)| \text{ for } T \le t \le S.$$

Finally, by (5) for $i = 0$ we have

$$U_0(y,T,t) = y(t) - y(T) \ge y(t) - |y(T)| = y(t) - |R_0(T)L_0 y(T)|$$

and this inequality completes the proof of the lemma.

3. Main results

Now we can prove some results which will be the generalization of the analogous ones given by Kusano and Onose in [1] and by Soltes in [3] for $n = 2$.

THEOREM 1. *Let*

$$\int_{t_0}^{\infty} R_{n-1}|a(t)|dt < \infty, \tag{8}$$

$$\int_{t_0}^{\infty} R_{n-1}|b(t)|dt < \infty. \tag{9}$$

Then every bounded oscillatory solution of the equation (1) tends to zero for $t \to \infty$.

PROOF. Let y be any bounded oscillatory solution of (1) on some $< T, \infty) \subset (t_0, \infty)$. Then multiplying the equation (1) by the function $R_{n-1}(t)$ and integrating it from T to $t \geq T$ and using the denotations (2) and (5) we get

$$U_{n-1}(y,T,t) = -\int_T^t R_{n-1}(s)a(s)f(y(g(s)))ds + \int_T^t R_{n-1}(s)b(s)ds \text{ for } t \geq T.$$

Then by (8) and (9) there exists $\lim U_{n-1}(y,T,t)$ for $t \to \infty$, because y is bounded. By Lemma 1 there exists $\lim y(t)$ (proper or improper) for $t \to \infty$, too. This limit must be equal to zero, since y is oscillatory.

THEOREM 2. *Let the conditions (6), (8) and (9) be fulfilled and*

$$g(t) \leq t \text{ for } t \geq t_0, \tag{10}$$

$$\limsup_{|y| \to \infty} \frac{f(y)}{y} < \infty. \tag{11}$$

Then every solution of the equation (1) is bounded.

PROOF. Let y be any solution of (1) on some $< T_0, \infty) \subset (t_0, \infty)$ Let us suppose, for contrary, that y is not bounded from above (the case of unboundness from belove is analogous). By (11) there exists $K > 0$ such that

$$\frac{f(z)}{z} \leq K \text{ for } z \geq 1. \tag{12}$$

By (8) ther exists $T \geq T_0$ such that

$$\int_T^{\infty} R_{n-1}(t)|a(t)|dt < \frac{1}{3K}. \tag{13}$$

By (9) there exists $M > 1$ such that

$$\int_T^{\infty} R_{n-1}(t)|b(t)|dt + \sum_{i=0}^{n-1} |R_i(T)L_i y(T)| \leq \frac{M}{3}. \tag{14}$$

Since the solution y is unbounded from above, there exists $S > T$ such that

$$y(S) \geq M \text{ and } |y(g(t))| \leq y(S) \text{ for } T \leq t \leq S. \tag{15}$$

Multiplying the equation (1) by $R_{n-1}(t)$, integrating it from T to $t \geq T$ and using the denotations (5) we get

$$U_{n-1}(y,T,t) = -\int_T^t R_{n-1}(s)a(s)f(y(g(s)))ds + \int_T^t R_{n-1}(s)b(s)ds \leq$$

$$\leq \int_T^t R_{n-1}(s)|a(s)f(y(g(s)))|ds + \int_T^t R_{n-1}(s)|b(s)|ds \text{ for } t \geq T.$$

By (15) and (13) from this inequality we have

$$U_{n-1}(y,T,t) \leq f(y(S)) \int_T^t R_{n-1}(s)|a(s)|ds + \int_T^t R_{n-1}(s)|b(s)|ds \leq$$

$$\leq f(y(S)) \int_T^\infty R_{n-1}(s)|a(s)|ds + \int_T^\infty R_{n-1}(s)|b(s)|ds \leq$$

$$\leq f(y(S))\frac{1}{3K} + \int_T^\infty R_{n-1}(s)|b(s)|ds \text{ for } T \leq t \leq S.$$

Applying Lemma 2 to this inequality and using (14) and (15) we obtain

$$y(t) \leq f(y(S))\frac{1}{3K} + \int_T^\infty R_{n-1}(s)|b(s)|ds + \sum_{i=0}^{n-1} |R_i(T)L_i y(T)| \leq$$

$$\leq f(y(S))\frac{1}{3K} + \frac{M}{3} \leq f(y(S))\frac{1}{3K} + \frac{y(S)}{3} \text{ for } T \leq t \leq S.$$

Finally, putting $t = S$ in this inequality and using (12) we obtain

$$y(S) \leq Ky(S)\frac{1}{3K} + \frac{y(S)}{3} = \frac{2}{3}y(S).$$

This contradiction completes the proof o the theorem.

THEOREM 3. *Let the assumptions of the Theorem 2 be fulfilled. Then every oscillatory solution of the equation (1) tends to zero for $t \to \infty$.*

PROOF. This theorem is direct consequence of the previous ones.

REMARK. For $n = 2$ the assumptions of the Theorem 3 have a form

$$\int_{t_0}^\infty \frac{dt}{r_1(t)} < \infty, \int_{t_0}^\infty R_1(t)|a(t)|dt < \infty \text{ and } \int_{t_0}^\infty R_1(t)|b(t)|dt < \infty,$$

which are weaker then those ones due to Kusano and Onose in [1]

$$\int_{t_0}^\infty \frac{dt}{r_1(t)} < \infty, \int_{t_0}^\infty |a(t)|dt < \infty \text{ and } \int_{t_0}^\infty |b(t)|dt < \infty.$$

For instance, the equation

$$(t^2y'(t))' + \frac{1}{t}y(t) = \frac{1}{t^2}\sin\ln t - \frac{1}{t}(\cos\ln t + \sin\ln t)$$

hase the oscillatory solution

$$y(t) = \frac{1}{t}\sin\ln t.$$

This equation fulfills the assumptions of the Theorem 3, but does not fulfill the conditions given in [1].

References

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