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ON SOME DECOMPOSITIONS TO SUBDIRECT PRODUCT

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The purpose of this work is to find external characterisations of the partial algebras subdirect decompositions.

For a start let us specify some terminology and notations.

Let $\mathcal{A} = (A, \{F_j / j \in J\})$ and $\mathcal{B} = (B, \{G_j / j \in J\})$ two τ - type partial algebras and let $f : A \to B$ an algebras homomorphism.

If $a \in A^{\tau(j)}$ and $f \circ a = (f(a_k))_{k<\tau(j)} \in D(G_j)$ it doesn't result that $a = (a_k)_{k<\tau(j)} \in D(F_j)$ (for partial algebras)

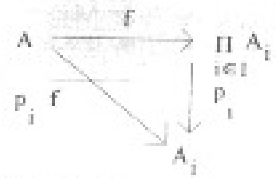
Definition 1. The $f : A \to B$ homomorphism is called full if $\forall j \in J$, $\forall a = (a_k)_{k<\tau(j)}$ we have $(f(a_k))_{k<\tau(j)} \in D(G_j)$ then that $\exists \overline{a} = (\overline{a}_k)_{k<\tau(j)} \in D(F_j)$, that $f \circ \overline{a} = f \circ a$, which means $f \circ \overline{a}_k = f \circ a_k$, for $k < \tau(j)$.

Let $(\mathcal{A}_i)_{i\in I} = (A_i, \{F_{ij} / j \in J\})$ a set of partial algebras of the same type τ. We consider $A = \prod_{i\in I} A_i$ (the direct product) and define $D(F_j) = \prod_{i\in I} D(F_{ij})$. We consider the canonical projections $p_i : \prod_{i\in I} A_i \to A_i$, $p_i(F_j(a)) = F_{ij}(p_i(a))$, $\forall a \in \prod_{i\in I} A_i$.

Observation. We have the equivalence p_i is full if and only if $\forall j \in J$, if $D(F_{ij}) \neq \emptyset$ for an arbitrary $i \in I$ then we have $D(F_{ij}) \neq \emptyset$ for each $i \in I$.

Definition 2. A decomposition of the $\mathcal{A}$ partial - algebra as a subdirect product of the set $\{\mathcal{A}_i / i \in I\}$ is a monomorphism $f : A \to \prod_{i\in I} A_i$ for which $p_i \circ f$ are full surjective homomorphisms.

If neither of the $p_i \circ f$ homomorphism are monomorphism the decomposition is called proper $p_i \circ f$.

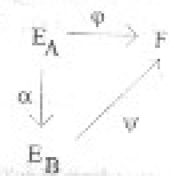


Let q_i be Ker $(p_i \circ f)$. Then, $a_i \equiv a / q_i$, and $\bigcap q_i = \Delta_A$.

The decomposition is proper if and only if $q_i \supset \Delta_A$ for every $i \in I$. The reciprocal theorem is also true. If $\mathcal{A}$ is partial algebra and $\{q_i / i \in I\}$ a set of congruences within $\mathcal{A}$ that $\bigcap q_i = \Delta_A$, then there exists a decomposition f of $\mathcal{A}$ as $\mathcal{A}$ subdirect product of the set $\{A / q_i / i \in I\}$ that $q_i = $ Ker $(p_i \circ f)$.

Let us consider the $\mathcal{K}$ category, the diagram scheme $\mathcal{M}$ and the category of the diagrams $F(\mathcal{M}, \mathcal{K})$. Let $A \in \mathrm{Ob}\,\mathcal{K}$. We consider the functor $E_A : \mathcal{M} \to \mathcal{K}$, $E_A(M) = A$, $\forall M \in \mathrm{Ob}\,\mathcal{M}$, $E_A(\alpha) = e_A$, $\forall \alpha \in \mathrm{Mor}\,\mathcal{M}$.

Definition 3. The functor $E_A : \mathcal{M} \to \mathcal{K}$, $A \in \mathrm{Ob}\,\mathcal{K}$ is called the inverse limit of the diagram $\mathcal{F} : \mathcal{M} \to \mathcal{K}$ if there exists a natural transformation $\varphi : E_A \to F$ so that for every natural transformation $\psi : E_B \to F$, $B \in \mathrm{Ob}\,\mathcal{K}$, there exists a unique natural transformation $\alpha : E_A \to E_B$ and $(E_A, \varphi) = \underleftarrow{\lim}\, F$.



Let $\mathcal{M}$ be a diagram scheme and $F : \mathcal{M} \to \mathrm{Alg}_p(\tau)$ a diagram of scheme $\mathcal{M}$. We suppose that $(\mathcal{A}, \varphi) = \underleftarrow{\lim}\, F$ and we consider the direct product $\prod_{M \in \mathrm{Ob}\,\mathcal{M}} F(M)$, with the canonical projections p_M.

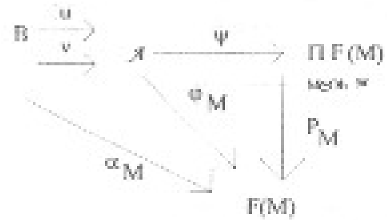
Because $(\mathcal{A}, \varphi) = \underleftarrow{\lim}\, F$, there exists a natural transformation φ_M.

The existence of ψ is proved by using the properties of the direct product $\prod_{M \in \mathrm{Ob}\,\mathcal{M}} F(M)$

We will now show that ψ is monomorphis, (which means $\psi u = \psi v \Rightarrow u = v$)

Let $B \overset{u}{\underset{v}{\rightrightarrows}} \mathcal{A}$, where $\psi u = \psi v$

Then $p_M(\psi u) = p_M(\psi v) \Rightarrow \varphi_M u = \varphi_M v = \alpha_M$.



and $\left.\begin{array}{l}\varphi_M u = \alpha_M \\ \varphi_M v = \alpha_M\end{array}\right\} \Rightarrow u = v$ (it is deducted from the definition of the inverse limit).

So, if we consider that φ_M are full surjective homomorphism then ψ is a decomposition to a subdirect product of the $\mathcal{A}$ algebra, which means $\lim\limits_{\leftarrow} F$ is a decomposition to a subdirect product.

We turn now to presenting the main result of this paper.

We might ask ourselves if there are any necessary and sufficient conditions for a decomposition in a subdirect product to be the inverse limit of a diagram. It is natural that we look for these conditions to be exposed in the congruence system's language which corresponds to such a decomposition.

Let $\mathcal{A} \in Alg_p(\tau)$ and $\mathcal{S} = \{q_t \in Con(\mathcal{A}) / t \in T\}$. We introduce the following notations:

$$q_{ts} = q_t \vee q_s,\ t, s \in T$$
$$q = \vee\{q_t / t \in T\}$$

We define the diagram schemes:

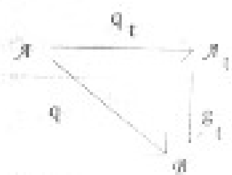
1. $\mathcal{T}$, $Ob\,\mathcal{T} = T \bigcup (T^2 - \Delta_T)$ and the only morphism which can be defined on $\mathcal{T}$ are those witch are identical and for each (t,s) pair, when $t \neq s$ there is only one morphism $t \xrightarrow{\tau_{ts}} (t,s)$.

We consider the diagram $D : \mathcal{T} \to Alg_p(\tau)$.

$D(t) = \mathcal{A} / q_t = \mathcal{A}_t$ $\quad D(t,s) = \mathcal{A} / q_{ts} = \mathcal{A}_{ts}$ $\quad D(\tau_{ts}) = g_{ts}$ defined through

2. $\overline{\mathcal{T}}$, $Ob\,\overline{\mathcal{T}} = T \bigcup \{\infty\}$, $\infty \notin T$, and all the morphism of $\overline{\mathcal{T}}$ are identical and for each t there is only one morphism $t \xrightarrow{\tau_t} \infty$. We consider the diagram $\overline{D} : \overline{\mathcal{T}} \to Alg_p(\tau)$

$\overline{D}(t) = \mathcal{A} / q_t = \mathcal{A}_t$ $\quad \overline{D}(\infty) = \mathcal{B} = \mathcal{A} / q$ $\quad \overline{D}(\tau_t) = g_t$ defined through



We define a set of properties for the S system:

Definition 4. S is weakly permutable if for every set $\{a_t / t \in T\}$ of elements from $\mathcal{A}$, if $\forall (t, s) \in T^2$, $a_t \equiv a_s(q_{ts}) \Rightarrow \exists a \in A$ that $a \equiv a_t(q_t)$, $t \in T$.

Definition 5. S is absolutely permutable if for every $\{\mathcal{A}_t / t \in T\}$ if

$$a_t \equiv a_s(q) \ \forall (t, s) \in T^2 \Rightarrow \exists a \in A, \ a \equiv a_t(q_t), \forall t \in T$$

Observation 1. absolutely permutable $\Rightarrow$ weakly permutable.

2. absolutely permutable $\Leftrightarrow$ weakly permutable and for every $q_t, q_s \in S$, $q_t \neq q_s, q_{ts} = q_t \vee q_s = q$

Theorem. Let $\varphi : \mathcal{A} \to \prod_{t \in T} \mathcal{A}_t$ a full subdirect decomposition if the corresponding system $S = \{q_t / t \in T\}$, $q_t = \mathrm{Ker}(p_t \varphi)$ is weakly permutable the $\mathcal{A}$ can be represented as the inverse limit of a $(\mathcal{A}, \varphi) = \varprojlim D$, D diagram, with $\psi_t = p_t \varphi$

If $(\mathcal{A}, \psi) = \varprojlim D$ and $\psi(t)$ is a full surjective homomorphism for each $t \in T$ then $\mathcal{A} \xrightarrow{\varphi} \prod_{t \in T} D(t)$ (with the p_t projections), $p_t \circ \varphi = \psi_t$ is a full subdirect decomposition of $\mathcal{A}$, having the corresponding system of congruences weakly permutable.

The theorem remains true when we replace "weakly permutable" with "absolutely permutable" and the diagram D with $\overline{D}$.

We have obtained external characterisations of some of the subdirect decompositions of the set $\{\mathcal{A}_t / t \in T\}$.

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