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A NOTE ON THE REDUCTION OF n-SEMIGROUPS OF FRACTIONS

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Abstract. The paper deals with n-semigroups of fractions and their binary reducts. The main result, contained in Proposition 3, states that if we perform on a semicommutative n-semigroup a binary reduction followed by a construction of a semigroup of fractions or we start by constructing an n-semigroup of fractions and then a binary reduction, we are conducted at isomorphic semigroups.

1. M.S.Pop and M.Câmpian [3] generalized the method of constructing semigroups of fractions to the case of semicommutative (not necessarily commutative) cancellative n-semigroups.

Let $(A,\circ)$ be an n-semigroup, i.e. a nonvoid set A on which an associative n-ary operation $\circ : A^n \to A$ is defined. An n-semigroup $(A,\circ)$ is called **semicommutative** if the following equality

(1) $(a_1, a_2, \ldots, a_{n-1}, a_n)_\circ = (a_n, a_2, \ldots, a_{n-1}, a_1)_\circ$ holds for each

$a_i \in A$, $i = 1, \ldots, n$.

In the sequel we shall use the notation $(a_1^n)_\circ$ instead of $(a_1, \ldots, a_n)_\circ$, and if k consecutive factors of the product coincide, we shall write $\overset{(k)}{a}$.

An n-semigroup $(A,\circ)$ is called **entropic** if

(2) $((a_{11}^{1n})_\circ,(a_{21}^{2n})_\circ,\ldots,(a_{n1}^{nn})_\circ)_\circ = ((a_{11}^{n1})_\circ,(a_{12}^{n2})_\circ,\ldots,(a_{1n}^{nn})_\circ)_\circ$ for each $a_{ij} \in A$, $i,j = 1,\ldots,n$

Any semicommutative n-semigroup is entropic, while the converse is not true.

An n-semigroup $(A,\circ)$ is called **right** (resp. **left**) cancellative with respect to $S \subseteq A$ if (for each a,b $\in$ A, $s_j \in S$, $j = 1,2,\ldots,n$)

(3) $$(a,s_2^n)_\circ = (b,s_2^n)_\circ \text{ implies } a = b$$

resp.

(3') $$(s_1^{n-1},a)_\circ = (s_1^{n-1},b)_\circ \text{ implies } a = b$$

By application of the associative law one obtains that a right and left cancellative n-semigroup (with respect to S) is cancellative (with respect to S) i.e. for i = 1, . . . ,n

(4) $$(s_1^{i-1},a_i,s_{i+1}^n)_\circ = (s_1^{i-1},b_i,s_{i+1}^n)_\circ \text{ implies } a_i = b_i$$

An ordered system $(u_1,\ldots,u_{n-1}) \in A^{n-1}$ (shortly u_1^{n-1}) of n-1 elements of an n-semigroup is called **right identity (left identity)** if for each $a \in A$

(5) $$(a,u_1^{n-1})_\circ = a \quad \left((u_1^{n-1},a)_\circ = a\right).$$

An n-semigroup $(A,\circ)$ in which the equation $(a_1^{i-1},x,a_{i+1}^n)_\circ = a_i$ has a unique solution, for each $i \in \{1,2,\ldots,n\}$ and $a_1,\ldots,a_n \in A$, is called **n-group**

The solution of the equation $\left(\overset{(n-1)}{a},x\right) = a$ is called **skew element (or querelement)** of a, and it is denoted by $\bar{a}$; $\overset{(i-1)}{a}\,\bar{a}\,\overset{(n-i-1)}{a}$ is a right and left identity in the n-group, for each $i \in \{1,2,\ldots,n\}$

Proposition 1 ([3]). If $(A,\circ)$ is a semicommutative n-semigroup, cancellative with respect to an n-subsemigroup S, then there exists an n-semigroup A_s with identity (as a system of n-1 elements) and an injective homomorphism $f: A \to A_s$

such that the skew element $\overline{f(s)}$ of $f(s) \in A_s$ exists for each $s \in S$.

The n-semigroup of fractions A_s has an universal property which determines it up to isomorphism:

Proposition 2 ([3]). If $(A, \circ)$ is a semicommutative n-semigroup, cancellative with respect to an n-subsemigroup S, A_s is the n-semigroup of fractions of A with denominators in S^{n-1} and $f : A \to A_s$ is the canonical homomorphism, then for every homomorphism $\alpha : A \to B$, where (B,[]) is a semicommutative n-semigroup with identity, having the property that $\alpha(s)$ has a skew element in B, for all $s \in S$, there exists a unique homomorphism $\beta : A_s \to B$ such that $\beta \circ f = \alpha$.

2. We shall prove now that if we start from a semicommutative n-semigroup and we construct an n-semigroup of fractions and then a binary reduction of it, or we perform the binary reduction followed by the construction of a semigroup of fractions, we are conducted to the same result.

Let $(A, \circ)$ be an n-groupoid and $u_1, \ldots, u_{n-2} \in A$. Define a binary operation "$\cdot$" on A by

(6) $$x \cdot y = (x, u_1^{n-2}, y)_\circ, \quad (\forall) x, y \in A.$$

$(A, \cdot)$ is called **the binary reduct** of A with respect to $u_1, \ldots, u_{n-2}$, and it is denoted by $red_{u_1^{n-2}}(A, \circ)$ ([2]).

If $(A, \circ)$ is a semicommutative n-semigroup then its binary reducts are commutative semigroups.

Proposition 3. Let $(A, \circ)$ be a semicommutative n-semigroup, cancellative with respect to an n-semigroup S and $(A_{S^{n-1}}, *)$ the n-semigroup of fractions of A with denominators in S^{n-1}. Let $u_1, \ldots, u_{n-2} \in S$ be (arbitrary) fixed elements of S and $(A, \cdot) = red_{u_1^{n-2}}(A, \circ)$. Then the semigroup of fractions of $(A, \cdot)$ with denominators in S is isomorphic to $red_{u_1, \ldots, u_{n-2}}(A_{S^{n-1}}, *)$, where

$$u_i = \frac{\left(u_i, \overset{(n-1)}{s}\right)}{\overset{(n-1)}{s}}, \quad i = \overline{1, n-2}.$$

Proof. The semigroup operation in $(A, \cdot)$ is defined by (6); S being a cancellative n-subsemigroup of $(A, \circ)$ it follows immediately that S is a cancellative subsemigroup of $(A, \cdot)$. Denote the semigroup of fractions of A with denominators in S by $(A_s, \cdot)$; the operation in $(A_s, \cdot)$ is defined by

$$\frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 \cdot a_2}{s_1 \cdot s_2}.$$

The n-ary operation in $(A_s, *)$ is defined by (see [3]).

(7) $$\left(\frac{a_1}{s_{12}^{1n}}, \frac{a_2}{s_{22}^{2n}}, \ldots, \frac{a_n}{s_{n2}^{nn}}\right)_* = \frac{(a_1^n)_\circ}{(s_{12}^{n2})_\circ (s_{13}^{n3})_\circ \ldots (s_{1n}^{nn})_\circ}$$

The binary operation in $red_{u_1, \ldots, u_{n-2}}(A_S, *)$, is defined by

(8) $$\frac{a}{s_2^n} * \frac{b}{t_2^n} = \left(\frac{a}{s_2^n}, \frac{\left(u_1, \overset{(n-1)}{s}\right)}{\overset{(n-1)}{s}}, \ldots, \frac{\left(u_{n-2}, \overset{(n-1)}{s}\right)}{\overset{(n-1)}{s}}, \frac{b}{t_2^n}\right)_*.$$

The mapping $\varphi: (A_s, \cdot) \to red_{u_1, \ldots, u_{n-2}}(A_{S^{n-1}}, *)$, $\varphi\left(\frac{a}{s}\right) = \frac{a}{u_1^{n-2} s}$ is an isomorphism of semigroups.

The definition of φ does not depend on the choice of representatives: indeed if $\frac{a}{s} = \frac{b}{t}$, then $a \cdot t = b \cdot s$, or $(a, u_1^{n-2}, t)_\circ = (b, u_1^{n-2}, s)_\circ$, which implies $\frac{a}{u_1^{n-2} s} = \frac{b}{u_1^{n-2} t}$ i.e. $\varphi\left(\frac{a}{s}\right) = \varphi\left(\frac{b}{t}\right)$.

φ is a homomorphism of semigroups: if $\frac{a}{u}, \frac{b}{t} \in A_s$ then

$$\varphi\left(\frac{a}{u}\cdot\frac{b}{t}\right)=\varphi\left(\frac{a\cdot b}{u\cdot t}\right)=\frac{a\cdot b}{u_1^{n-2}(u\cdot t)}=\frac{(a,u_1^{n-2},b)_\circ}{u_1^{n-2}(u,u_1^{n-2},t)_\circ}$$ and

$$\varphi\left(\frac{a}{u}\right)*\varphi\left(\frac{b}{t}\right)=\frac{a}{u_1^{n-2}u}*\frac{b}{u_1^{n-2}t}=$$

$$=\left(\frac{a}{u_1^{n-2}u},\frac{\left(u_1,\overset{(n-1)}{s}\right)_\circ}{\overset{(n-1)}{s}},\ldots,\frac{\left(u_{n-2},\overset{(n-1)}{s}\right)_\circ}{\overset{(n-1)}{s}},\frac{b}{u_1^{n-2}t}\right)_*=$$

$$=\frac{\left(a,u_1,\overset{(n-1)}{s},u_2,\overset{(n-1)}{s},\ldots,u_{n-2},\overset{(n-1)}{s},b\right)_\circ}{\left(\left(u_1,\overset{(n-2)}{s},u_1\right)_\circ,\left(u_2,\overset{(n-2)}{s},u_2\right)_\circ,\ldots,\left(u_{n-2},\overset{(n-2)}{s},u_{n-2}\right)_\circ,\left(u,\overset{(n-2)}{s},t\right)_\circ\right)_\circ}$$

By using semicommutativity (and entropy) of "$\circ$" we have

$$\left((a,u_1^{n-2},b)_\circ,\left(u_1,\overset{(n-2)}{s},u_1\right)_\circ,\ldots,\left(u_{n-2},\overset{(n-2)}{s},u_{n-2}\right)_\circ,\left(u,\overset{(n-2)}{s},t\right)_\circ\right)_\circ=$$

$$=\left((a,u_1^{n-2},u)_\circ,\left(u_1,\overset{(n-1)}{s}\right)_\circ,\ldots,\left(u_{n-2},\overset{(n-1)}{s}\right)_\circ,(b,u_1^{n-2},t)_\circ\right)_\circ=$$

$$=\left(a,u_1,\overset{(n-1)}{s},\ldots,u_{n-2},\overset{(n-1)}{s},b,u_1^{n-2},u,u_1^{n-2},t\right)_\circ$$ which shows that

$$\varphi\left(\frac{a}{u}\cdot\frac{b}{t}\right)=\varphi\left(\frac{a}{u}\right)*\varphi\left(\frac{b}{t}\right).$$

It is easy to proof that φ is injective and surjective, and so φ is an isomorphism of semigroups.

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