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A NOTE ON THE DIRECT PRODUCT OF SEMICOMMUTATIVE n-GROUPS

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For the commutative finitely generated n-groups, Timm [4] generalized the theorem of the "basis" from the binary case, concerning the representation of an abelian finitely generated group as direct product of cyclic groups. In this note we show that the result (obtained of Timm) is also preserved for semicommutative finitely generated n-groups.

1.Preliminaries

The pair $(A,\circ)$,where A is a non-empty set, $\circ: A^n \to A$, $n \geq 2$, $n \in \mathbb{N}$ is an n-ary associative operation [1] is called an n-semigroup. An n-group is an n-semigroup $(A,\circ)$ in which the equations $(a_1,\ldots a_{i-1},x,a_{i+1},\ldots a_n)=a$, shortly $(a_1^{i-1},x,a_{i+1}^n)=a$, have unique solution in A, for arbitrary $a_1,\ldots,a_n \in A$, and for each $i \in \{1,\ldots n\}$. An n-semigroup (n-group) $(A,\circ)$ is commutative if the operation "$\circ$" is invariant under each permutation of the elements involved.

$(A, \circ)$ is semicommutative if $(a_1, a_2^{n-1}, a_n)_\circ = (a_n, a_2^{n-1}, a_1)_\circ$ for arbitrary $a_1, \dots, a_n \in A$. Obviously, for $n=2$ the commutative and semicommutative n-semigroup (n-group) concepts coincide. An n-semigroup (n-group) $(A, \circ)$ is entropic if

$$((a_{11}^{1n})_\circ, (a_{21}^{2n})_\circ, \dots, (a_{n1}^{nn})_\circ)_\circ = ((a_{11}^{n1})_\circ, (a_{12}^{n2})_\circ, \dots, (a_{1n}^{nn})_\circ)_\circ \qquad (1)$$

for arbitrary $a_{ij} \in A$; $i, j \in \{1, \dots, n\}$.

If $(A, \circ)$ is a semicommutative n-semigroup then it is an entropic n-semigroup. The converse is not true, but for n-groups the two notions are equivalent. In the n-group $(A, \circ)$, the solution of the equation

$(a, \dots, a, x)_\circ = a$, shortly $(\overset{(n-1)}{a}, x)_\circ = a$ is called the "querelement " of a [1] and it is denoted by $\overline{a}$. The element $\overline{a}$ has the additional property $(x, a, \dots, \overline{a}, \dots, a) = (a, \dots, \overline{a}, \dots, a, x) = x$ for each $x \in A$.

An n-group $(A, \circ)$ is called finitely generated if there are k fixed elements $a_1, \dots, a_k \in A$ such that every element of A is written as product of these elements and of their querelements, i. e. $A = <a_1, \dots, a_k>$.

If $k=1$, then the n-group $(A, \circ)$ is called cyclic and it is denoted by $<a_1>$.

If $(A, \circ)$ is an n-group, $a \in A$ a fixed element and $\cdot : A^2 \to A$, the binary operation defined by

$$x \cdot y = (x, \overset{(n-3)}{a}, \overline{a}, y)_\circ , \qquad (2)$$

then $(A, \cdot)$ is a group with the unit element a, denoted $red_a(A, \circ)$, called the reduced group with respect to $a \in A$. The reduced group of a semicommutative n-group is a commutative group.

If (A,·) is a group, $f \in Aut(A,\cdot)$ such that f^{n-1} is an inner automorphism of A generated by $a \in A$, $f(a) = a$; (3)

then the n-ary operation $* : A^n \to A$ defined by

$$(x_1^n)_* = x_1 \cdot f(x_2) \cdot f^2(x_3) \cdot \ldots \cdot f^{n-1}(x_n) \cdot a \qquad (4)$$

is an n-group operation and (A, *) is called the n-ary extension of the bigroup (A,·)[2] relatively to $a \in A$ and f, denoted by $ext_{f,a}(A,\cdot)$; moreover, $red_a(ext_{f,a}(A,\cdot))$ is isomorphic to $(A,\cdot)$. The extension of a commutative group by using an automorphism $f : A \to A$ and an element $a \in A$, such that f^{n-1} is the identity map and $f(a) = a$, is a semicommutative n-group.

If (A,∘) is an n-group, $f : A \to A$, $f(x) = \left(a, x, \overset{(n-3)}{a}, \bar{a} \right)$, then

$$ext^n_{f,a(n)}(red_a(A,\circ),\cdot) = (A,\circ). \qquad (5)$$

2. Let $(A_i, \circ_i)_{i \in I}$ be a family of n-groups. On the Cartesian product $\prod_{i \in I} A_i$ we define an n-ary operation "∘" in the following way:

$$\forall a_{ij} \in A_i; j = \overline{1,n}; i \in I \quad ((a_{i1})_{i \in I}, \ldots, (a_{in})_{i \in I})_\circ = ((a_{i1}, \ldots, a_{in})_{\circ_i})_{i \in I}$$

The algebra $\left(\prod_{i \in I} A_i, \circ \right)$ is called the direct product of the n-groups $(A_i, \circ_i)_{i \in I}$.

n-Groups form a variety and for this reason they have the properties of varieties and in the same time they have certain specific properties, too:

- The direct product $\left(\prod_{i \in I} A_i, \circ \right)$ of a family of (semicommutative or commutative) n-groups is again an n-group of the same type. For every $j \in I$, the canonical projection $p_j : \prod_{i \in I} A_i \to A_j$; $p_j((a_i)_{i \in I}) = a_j$ is a surjective homomorphism. If $(a_i)_{i \in I} \in \prod_{i \in I} A_i$, then $\overline{(a_i)_{i \in I}} = (\overline{a_i})_{i \in I}$;

- The direct product of a family of n-groups contains sub-n-groups which are isomorphic to the factors if and only if it is possible to define a homomorphism of n-groups between each two factors (this is a non - trivial affirmation, as the category of n-groups is not a connected one).

The connection between the direct product of n-groups and that of bigroups is given by the following

Theorem 1 [3]. **a)** *The direct product of the reduces of a family of n-groups* $(A_i, \circ_i)_{i \in I}$ *relatively to the elements* $a_i \in A_i$, $i \in I$ *is equal to the reduce of the direct product of the family relatively to* $(a_i)_{i \in I}$, *i.e.*

$$red_{(a_i)_{i \in I}}\left(\prod_{i \in I} A_i, \circ\right) = \prod_{i \in I} red_{a_i}(A_i, \circ_i) \qquad (6)$$

b) *If* $(A_i, \cdot_i)$ *are groups,* $f_i \in Aut(A_i, \cdot_i)$ *such that* f_i^{n-1} *is an inner automorphism of* A_i, *generating* $a_i \in A_i$, *when* $f_i(a_i) = a_i$, $i \in I$, *then for their n-ary extensions we have*

$$ext^{n}_{h,(a_i)_{i \in I}}\left(\prod_{i \in I} A_i, \cdot\right) = \prod_{i \in I} ext^{n}_{f_i, a_i}(A_i, \cdot_i) \qquad (7)$$

where $h: \prod_{i \in I} A_i \to \prod_{i \in I} A_i$; $h((x_i)_{i \in I}) = (f_i(x_i))_{i \in I}$.

For the commutative finitely generated n-groups, Timm [4] generalized the theorem of the "basis" from the binary case, concerning the representation of an abelian finitely generated group as direct product of cyclic groups.

We show that the result obtained of Timm is also preserved for semicommutative finitely generated n-groups, as follows.

Theorem 2. *If a semicommutative n-group is finitely generated, then it is isomorphic to a direct product of semicommutative n-groups whose binary reduces are cyclic groups.*

Proof. Let $(A, \circ)$ be a finitely generated semicommutative n-group and $a \in A$, one of the generators. The reduced group $red_a(A, \circ) = (A, \cdot)$ is a commutative finitely generated group; from the "basis" theorem it is isomorphic to a direct product of cyclic groups $(A_i,\cdot)$, $i=\overline{1,k}$, i.e. there is an isomorphism

$\varphi : red_a A \to \prod_{i=1}^{k} A_i$.

Let us denote $x_j = (p_j \circ \varphi)(x)$, $x \in A$, $a_j = (p_j \circ \varphi)(a)$, where p_j is the canonical projection of $\prod_{i=1}^{k} A_i$ onto A_j, $j=\overline{1,k}$; therefore $\varphi(a) = (a_1, \ldots, a_k)$.

If $f : A \to A$, $f(x) = (a, x, \overset{(n-3)}{a}, \bar{a})$, then by (5) we have $(A,\circ) = ext^n_{f,a^{[1]}}(red_a(A,\circ),\cdot)$ and φ is an isomorphism for the n-ary extensions too:

$$ext^n_{f,a^{[1]}}(red_a(A,\circ),\cdot) \cong ext^n_{h,\varphi(a^{[1]})}\left(\prod_{i=1}^{k} A_i,\cdot\right) \tag{8}$$

where $h = \varphi \circ f \circ \varphi^{-1}$ satisfies the conditions (3).

Let $f_i \in Aut(A_i,\cdot)$ be such that $f_i(c_i) = c_i$; $f_i^{\,n-1} = 1_{A_i}$ where $c = a^{[1]}$ and $(p_i \circ \varphi)(c) = c_i$, $i=\overline{1,k}$ and $h = (f_i)_{i=\overline{1,k}}$ .Denote $ext_{f_i,c_i}(A_i,\cdot) = (A_i,\circ_i)$ then $f_i(x_i) = (a_i, x_i, \overset{(n-3)}{a_i}, \bar{a}_i)_{\circ_i}$ .Now by Th.1 h) we have

$$ext^n_{h,\varphi(c)}\left(\prod_{i=1}^{k} A_i\right) = \prod_{i=1}^{k} ext^n_{f_i,c_i}(A_i,\cdot) \text{ and } red_{a_i}(ext^n_{f_i,c_i}(A_i,\cdot))$$

are cyclic groups.

In particular, for the commutative case we have $f = 1_A$ and we get Timm's results 6.5. [4].

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