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SUBGROUPS AND QUOTIENT GROUPS OF ABELIAN GROUPS WITH D.S.I.P.

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Abstract: It is known that if an abelian group A has the direct summand intersection property (for short D.S.I.P.), then any direct summand B, of A, and A/B have the same property. In this work we will study necessary and/or sufficient conditions for which some subgroups of group A (with D.S.I.P.), which are not direct summands and the quotient gruops corresponding to, to have D.S.I.P.. Thus, being given an abelian group A, with D.S.I.P. and m a non-null natural number, we will study, here, the following subgroups: mA, A[m], $m^{-1}A$ (in this case, A is a subgroup of a group G), F(A)-the Frattini subgroup of A and B_A-the p-basic subgroup of A, for a p-any prime number, as well as the quotient corresponding groups. All through this paper by group we mean abelian group in additive notation and we will note with **P** the set of all prime numbers.

1. The subgroups of form mA, m>0

We remind that a group A has the direct summand intersection property (for short D.S.I.P.), if intersection of any two direct summands of A is again a direct summand in A.

We begin our investigations with the following elementary result.

Remark 1.1.: If A is an elementary group, then any subgroup B, of A, and A/B have D.S.I.P.. In particular, for every $m \in N^*$, mA and A/mA have D.S.I.P..

Proof: According to [7,&7] and [19,3.1.] any subgroup B of an elementary group A is a direct summand in A; so B and A/B have P.I.S.D. (see [19,2.6.])

Further on we will study the necessary and/or sufficient conditions for which the subgroup mA, m being a natural number, should have D.S.I.P., if A has this property.

Proposition 1.2.: *Let A be a abelian group and $m \in N^*$. Then the following statements occur:*

a) If B is a direct summand of A, then mB is a direct summand of mA. If A[m]=0, then the converse occurs, that is: if mB is a direct summand in mA, then B is a direct summand in A.

b) If B is a subgroup of mA, then there is a C subgroup of A such that B=mC.

c) If A[m]=0, then $mT \cap mS = m(T \cap S)$, for every T and S subgroups of A.

Proof: a) If B is a direct summand in A, then $A=B\oplus C$, with C subgroup of A. So for every $a \in A$, there is uniquely a $b \in B$ and $c \in C$ such that a=b+c. It follows that ma=mb+mc and $mA \subseteq mB+mC$. Since $mB+mC \subseteq mA$, it follows that mA=mB+mC and since $mB \cap mC \subseteq B \cap C=0$, it follows that $mA=mB\oplus mC$.

Conversely, let $mA=mB\oplus mC$ be a direct decomposition of mA. Then for every $a \in A$, there is uniquely a $b \in B$ and a $c \in C$ such that ma=mb+mc. So m(a-b-c)=0. Since A[m]=0, it follows that a=b+c. If $x \in B \cap C$, then $mx \in mB \cap mC=0$ and so x=0. Hence $A=B\oplus C$.

b) Let B be a subgroup of mA and $C=\{a \in A | ma \in B\}$. Then B=mC.

c) Let T and S be two subgroups of A and $x \in mT \cap mS$. Then x=mt=ms, with t∈T and s∈S. It follows that m(t-s)=0. Since A[m]=0, we obtain that t=s, so $x \in m(T \cap S)$. Hence $mT \cap mS \leq m(T \cap S)$, and since $m(T \cap S) \leq mT \cap mS$, we obtain the statement equality.

Theorem 1.3.: *Let A be an abelian group, with A[m]=0, where $m \in N^*$. Then the following statements are equivalent:*

a) A has D.S.I.P.;

b) mA has D.S.I.P..

Proof: a)⇒b) Let T and S be two direct summands in mA. According to (1.2.) there is B and C direct summands in A, such that T=mB and S=mC. Again from (1.2.) it follows that $T \cap S = mB \cap mC = m(B \cap C)$ is a direct summand in mA. So mA has D.S.I.P..

b)⇒a) Let B and C be two direct summands in A. Then mB, mC and $m(B \cap C) = mB \cap mC$ are direct summands in mA (see (1.2.)a),c)). Now (1.2.)a) completes the demonstration.

Remark 1.4.: In the enunciation of Theorem (1.3.), the condition A[m]=0 is absolutely necessary. We are going to demonstrate that if p is a prime number, there are groups with the property that A[p]≠0, pA have D.S.I.P. and A doesn't have this property anymore. Indeed, let $A = Z(p) \oplus Z(p^3)$ be a torsion group. According to [13,Theorem 2.], this group has not D.S.I.P., but $pA = pZ(p^3) \cong Z(p)$ has this property (it is obvious that A[p]≠0).

Corollary 1.5.: *Let A be a torsion-free group. Then A has D.S.I.P. if and only if mA has the same property, for every $m \in N^*$.*

Proof: If A is a torsion-free group, then for every $m \in N^*$, A[m]=0. Now we can apply the Theorem (1.3.).

Corollary 1.6.: *Let A be a torsion-free group. Then the following statements occur:*

a) If A is a free group (or a countable free group or a free abelian group of the power of the continuum), then for every $m \in N^*$, mA has D.S.I.P..

b) If A is a W-group (that is a Whitehead group) or a subgroup of W-group, or a direct sum of W-groups, then mA has D.S.I.P., for every $m \in N^*$.

c) If A is a completely decomposable homogeneous group, then for every $m \in N^*$, mA has D.S.I.P..

d) If A is not reduced, then A has D.S.I.P. if and only if mB has the same property, for every $m \in N^*$, where B is the reduced part of A.

e) If A is indecomposable, of finite rank, then for every index set I and every $m \in N^*$, the groups $\bigoplus_I mA$ have D.S.I.P. and the ring of endomorphism of mA is semi-hereditary right.

f) The group A has D.S.I.P. if and only if the pure subgroups of mA coincide with the direct summands of mA, for every $m \in N^*$.

Proof: a) The enunciation results from [19,2.2.] (respectively [19, 2.4.]) and (1.5.).

b) We can apply [19,2.8.] and (1.5.).

c) According to [13,Theorem 5.], A has D.S.I.P.. Now we can apply (1.5.).

d) Let A=D⊕B be a direct decomposition of group A, with D-divisible and B reduced. According to [19,5.12.], A has D.S.I.P. if and only if B has this property. And again (1.5.) completes the proof.

e) The enunciation results from [11,Theorem 4.1.], (1.1.) and (1.5.).

f) The pure subgroups of mA coincide with the direct summands of mA if and only if mA has D.S.I.P. (see [19,5.17.]). According to (1.5.), this is equivalent with the fact that A has D.S.I.P.

Corollary 1.7.: *Let A be a p-group. The following statements are equivalent.*

a) A has D.S.I.P.;

b) For every $m \in N^*$, mA has D.S.I.P..

Proof: Because the implication b)$\Rightarrow$a) is evident (we consider m=1), we are going to demonstrate only a)$\Rightarrow$b). So, let A be a p-group with D.S.I.P. and let m be any natural number. According to [13, Theorem 2.], we have two cases.

Case I. There is $n \in N^*$, such that $A=Z(p^n)$. If $m=p^i, i \geq 1$, then:

$$mA = \begin{cases} p^i Z(p^n) \cong Z(p^{n-i}), if\ i \leq n-1 \\ 0, if\ i \geq n \end{cases}$$

It follows that, in this case, mA is idecomposable and so mA has D.S.I.P.. If $m=p^i q$, $i \geq 0$, (p,q)=1, then:

$$mA = q(p^i Z(p^n)) = \begin{cases} qZ(p^{n-i}), if\ i \leq n-1 \\ 0, if\ i \geq n \end{cases} = \begin{cases} Z(p^{n-i}), if\ i \leq n-1 \\ 0, if\ i \geq n \end{cases}$$

In this case $Z(p^{n-i})[q]=0$ (in fact, in general, if a, $b \in N^*$, and (a,b)=1, then Z(a)[b]=0). Now (1.3.) completes the proof.

Case II. $A = B_p \oplus C_p$, with $pB_p=0$, $C_p=0$ or $C_p=Z(p^n)$. If p^k divides m, $k \geq 1$, then $mA=C_p$ has D.S.I.P.. If for every $k \geq 1$, p^k does not divide m, then $A[m]=B_p[m] \oplus C_p[m]=0$ (see (2.1.)a). And again (1.3.) completes the proof.

Corollary 1.8.: *Let A be a torsion group. The following statements are equivalent.*

a) A has D.S.I.P.;

b) For every $m \in N^*$, mA has D.S.I.P..

Proof: Similar to (1.7.) it is sufficient to prove that a)$\Rightarrow$b). Let A be a torsion group with D.S.I.P.. According to [19,3.3.], $A = \left(\bigoplus_{p \in P_1} A_p\right) \oplus \left(\bigoplus_{p \in P_2} B_p\right) \oplus \left(\bigoplus_{p \in P_2} C_p\right)$, where: $P_1 \subseteq P$, for every $p \in P \setminus P_2$, A_p is a idecomposable p-group, and for every $p \in P \setminus P_1$, $B_p = \bigoplus_{m_p} Z(p)$ $(m_p \in N^*)$, $C_p=0$ or $C_p=Z(p^n)$. Then $mA = \left(\bigoplus_{p \in P_1} mA_p\right) \oplus \left(\bigoplus_{p \in P_2} mB_p\right) \oplus \left(\bigoplus_{p \in P_2} mC_p\right)$, according to (1.2.)a). According to (1.7.), every direct summand from the above decomposition of mA has D.S.I.P. and is fully invariant (see [7,&2]). Now [13,Lemma 1.] completes the proof.

Remark 1.9.: In (1.7.) and (1.8.) the universal quantifier $\forall$ has a fundamental role. Its absence might imply the absence of the implication b)$\Rightarrow$a). For example: if $A=Z(4) \oplus Z(4)$, then $2A \cong Z(2) \oplus Z(2)$ has D.S.I.P., and A doesn't have this property anymore.

According to [19,6.4.] a splitting mixed group with D.S.I.P. takes the form:

$$(1) \quad A=\left(\bigoplus_{p\in P_1} A_p\right)\oplus\left(\bigoplus_{p\in P_2}\left(\bigoplus_{m_p} Z(p)\right)\right)\oplus\left(\bigoplus_{m_0} Q\right)\oplus B,$$

where P_1 and P_2 are subsets of the set P of all prime numbers and $P_1 \cap P_2 = \emptyset$; A_p is reduced, idecomposable, for every $p \in P_1$, and B is reduced torsion-free with D.S.I.P. (if $m_0 = 0$, then B is completely decomposable, homogeneous, of finite rank). If B is fully invariant in A, then the converse is true, that is every group of form (1) has D.S.I.P.. Combining these results with what we obtain in (1.5.) and (1.8.) we have:

Corollary 1.10.: *If the direct summand B from the (1) decomposition is fully invariant, then the group A (from the (1) decomposition), has D.S.I.P., if and only if for every $m \in N^*$, mB has D.S.I.P..*

Let $A/mA = \bigoplus_{p|m}(A/mA)_p$ be the direct decomposition of A/mA in its p-subgroups, according to [7,8.4.]. From [13,Theorem 1.], it follows that A/mA has D.S.I.P. if and only if for every $p \in P$ with $p|m$, $(A/mA)_p$ is a p-group with D.S.I.P.. Thus we have the following theorem:

Theorem 1.11.: *Let A be an abelian group with D.S.I.P. and let m be any natural number. In any of the following situations, A/mA has D.S.I.P.:*

a) A is a p-group,

b) $A = \bigoplus_{p\in P} A_p$ is a torsion group, decomposed according to [7,8.4.], and $\bigoplus_{p|m} m A_p = 0$,

c) $A = Z$,

d) A is divisible,

e) m is a prime number.

Proof: a) Let A be a p-group with D.S.I.P.. If $A = Z(p^n)$, with $n \in N^*$ and p^i divides m ($i \ge 1$), then, from (1.7.) it follows that $A/mA \cong Z(p^n)/Z(p^{n-i}) \cong Z(p^i)$. If p^i does not divide m, then, again from (1.7.) it follows that $A/mA \cong A$. So A/mA has D.S.I.P.. Now we suppose that $A = B_p \oplus C_p$, where $B_p = \bigoplus_{m_p} Z(p)$, and $C_p = 0$ or $C_p = Z(p^2)$. Then either $mA = C_p$, if p divides m ($i \ge 1$), or $mA = A$, if p does not divide it. It follows that A/mA has D.S.I.P..

b) Let $A = \bigoplus_{p\in P} A_p$ be a torsion group with D.S.I.P.. According to hypothesis and to (1.2.(a)) $mA = \bigoplus_{p\in P_0} A_p$, where $P_0 = \{p \in P \mid p \text{ does not divide } m\}$, and so A/mA has D.S.I.P..

c) Let $m = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$ be the decomposition of m in the product of power of prime numbers ($p_i \in P$, $n_i \in N^*$, $\forall i = \overline{1,k}$, $k \in N^*$), and $A = Z$. Then $A/mA = Z/mZ = Z(m) = Z(p_1^{n_1}) \oplus Z(p_2^{n_2}) \oplus \ldots \oplus Z(p_k^{n_k})$, and it has D.S.I.P., according to [19,3.3.].

d) If A is divisible, then mA=A and A/mA=0

e) If p is a prime number, then A/pA is an elementary p-group, and in this case it completes [19,2.6] the demonstration.

2. The subgroups of form A[m], m>0

We remind that if A is any group and m is a not-null natural number, then $A[m] = \{a \in A | ma = 0\}$.

Proposition 2.1.: *Let A be an abelian group and $m \in N^*$. Then the following statements occur:*
a) If B is a direct summand in A, then B[m] is a direct summand in A[m];
b) If T and S are subgroups in A, then $T[m] \cap S[m] = (T \cap S)[m]$;
c) If A any pure subgroup is a direct summand (in A), and C is a direct summand in A[m], then there is B-direct summand in A, such that B[m] = C.

Proof: Let $A = B \oplus C$ be any direct decomposition of A. We are going to demonstrate that: $A[m] = B[m] \oplus C[m]$. If $a \in A[m]$, then there is $b \in B$ and there is $c \in C$, such that a=b+c and ma=mb+mc=0. Since $B \cap C = 0$, it follows that mb=m(-c)=-mc=0; that is $b \in B[m]$ and $c \in C[m]$. Thus $A[m] \subseteq B[m] \oplus C[m]$ because $B[m] \cap C[m] \subseteq B \cap C = 0$. Since $B[m] + C[m] = B[m] \oplus C[m] \subseteq A[m]$, we obtain the statement equality.
b) Let be $x \in T[m] \cap S[m]$. Then $x \in T$ with mx=0, and $x \in S$ with mx=0. It follows that $x \in (T \cap S)[m]$ and so $T[m] \cap S[m] \subseteq (T \cap S)[m]$. Since $(T \cap S)[m] \subseteq T[m] \cap S[m]$, it follows that the statement equality occurs.
c) If mA=0, then A[m]=A and any direct summand C of A[m] is a direct summand in A, and C[m]=C. If $mA \neq 0$, then let be $a \in A$, with $\langle a \rangle[m] = 0$, and $B = \langle C, a \rangle$, where C is a subgroup of $A[m]$. We are going to show that B[m]=C. If $b \in B$, then b=c+na, with $c \in C$ and $n \in Z$. The equality mb=0, involves m(na)=0; so a=0 and B[m]=C. If B[m] is a direct summand in A[m], then from [7,29.1.], it follows that B is a pure subgroup in A. According to hypothesis and to [7,27.5.], B is a direct summand in A.

From (2.1.)a) it results that the problem of determining the groups A with D.S.I.P., for which A[m] has D.S.I.P., confines itself to determine the groups for which $A[m] \neq 0$.

Theorem 2.2.: *Let A be a torsion group. If A has D.S.I.P., then for every $m \in N^*$, A[m] also has D.S.I.P.*

Proof: Let A be a torsion group with D.S.I.P. and let $A = \bigoplus_p A_p$ be the direct decomposition of A in its p-subgroups, according to [7,8.4.]. Then $A[m] = \bigoplus_p A_p[m]$, according to (2.1.)a). We are going to demonstrate that for every p-prime number, $A_p[m]$ has D.S.I.P. Since A_p is a p-group with D.S.I.P., we have two cases:

Case I: A_p is a reduced idecomposable group. In this case there is $n \in N^*$, such that $A_p = Z(p^n)$ and for every $m \in N^*$, $A_p[m]$ has D.S.I.P.

Case II: $A_p = B_p \oplus C_p$, where $B_p = \bigoplus_{m_p} Z(p)$, and $C_p = 0$ or $C_p = Z(p^\infty)$. In this case, for every pure subgroup G of A_p, $pG = G \cap pA_p = G \cap pC_p = C_p$. If $C_p = 0$, then G is a direct summand in A_p, according to [7,27.5.], and if $C_p = Z(p^\infty)$, then G contains or does not contain $Z(p^\infty)$ (because $Z(p^\infty)$ has not pure subgroups except for the trivial ones-see [7,26 (c).]). So pG=0 or pG= $Z(p^\infty)$, in both cases, G is a direct summand in A_p. Now let T and S be two direct summands in $A_p[m]$. According to (2.1.)c) there are U and V-direct summands in A_p, such that U[m]=T and V[m]=S. Since A_p has D.S.I.P., it follows that $U \cap V$ is a direct summand in A_p. From (2.1.)a) and (2.1.)b) it follows that $(U \cap V)[m]=U[m] \cap V[m]=T \cap S$ is a direct summand of $A_p[m]$. Hence $A_p[m]$ has D.S.I.P.. Since the subgroups $A_p[m]$, $p \in P$, are fully invariants, [13,Lemma 1.] completes the demonstration.

Corollary 2.3.: *If A is a splitting mixed group with D.S.I.P., then for every $m \in N^*$, A[m] has D.S.I.P..*

Proof: If the group A is as in the statement, then $A=\left(\bigoplus_p A_p\right) \oplus \left(\bigoplus_{m_0} Q\right) \oplus B$ (see [19,6.4.]), where A_p is a reduced p-group with D.S.I.P., and B is a reduced torsion-free group with D.S.I.P. (if $m_0 \neq 0$, then B is completely decomposable, homogeneous, of finite rank). It follows that $A[m]=\bigoplus_p A_p[m]$ and now (2.2.) completes the demonstration.

Corollary 2.4.: *Let A be an abelian group with D.S.I.P. and $m \in N^*$. In any of the following situations, A[m] has D.S.I.P.:*

a) A is a torsion group,

b) A is a divisible group,

c) A is a torsion-free group,

d) A is a splitting mixed group,

e) The subgroup T(A)-the torsion part of A, is either bounded or finitely generated, and $(T(A))^1=0$ (see [19,6.6.]).

f) A is a mixed group and $A \oplus A$ has D.S.I.P..

Remark 2.5.: The converse of (2.2.) is generally false. Thus, there is $m \in N^*$ with the property that A[m] has D.S.I.P., but A doesn't have this property anymore. For example, let p be a prime number and let be $A=Z(p^2) \oplus Z(p^3)$. Then $A[p] \cong Z(p) \oplus Z(p)$. According to [13,Theorem 2.], A[p] has D.S.I.P., and A doesn't have D.S.I.P..

We close this paragraph with the following result:

Proposition 2.6.: *For every $m \in N^*$, the group A/A[m] has D.S.I.P. if and only if mA has this property.*

Proof: Let m be a not-null natural number and $\rho_m : A \to A$ stands for the multiplication by m in A. Then the kernel $\ker \rho_m$ of ρ_m is A[m], and the image $\operatorname{Im} \rho_m$ of ρ_m is mA. It follows that: $0 \to A[m] \xrightarrow{i} A \xrightarrow{\rho_m} mA \to 0$ is an exact sequence, and so $A/A[m] \cong mA$.

3. The subgroups of form $m^{-1}A, m>0$

Let G be a abelian group, A a subgroup of G and $m \in N^*$, $m \geq 2$. It is known that $m^{-1}A = \{g \in G | mg \in A\}$ is a subgroup of G. In this paragraph we will see under what conditions $m^{-1}A$ has D.S.I.P., if A has this property.

Proposition 3.1.: *If A is a subgroup of G, $m \in N^*$, and B and C are two subgroups of A, then the following statements occur:*

a) $m^{-1}B \cap m^{-1}C = m^{-1}(B \cap C)$.

b) If $G[m]=0$ and T is a subgroup in $m^{-1}A$, then there is an S-subgroup of G, such that $T = m^{-1}S$.

Proof: a) Let be $x \in m^{-1}B \cap m^{-1}C$. Then $mx \in B$ and $mx \in C$; so $mx \in B \cap C$ and $x \in m^{-1}(B \cap C)$. It follows that $m^{-1}B \cap m^{-1}C \subseteq m^{-1}(B \cap C)$ (1). If $y \in m^{-1}(B \cap C)$, then $my \in B \cap C$, so $my \in B$ and $my \in C$, that is $y \in m^{-1}B \cap m^{-1}C$. Hence $m^{-1}(B \cap C) \subseteq m^{-1}B \cap m^{-1}C$ (2). From relationships (1) and (2) we obtain the statement equality.

b) Let be $S = \{mt | t \in T\} = mT$. Then, according to [7,&1] we have $m^{-1}S) = m^{-1}(mT) = T + A[m] = T$, because $A[m] \leq G[m] = 0$.

Theorem 3.2.: *Let G be an abelian group with $G[m]=0$, for a $m \in N^*$, and A a pure subgroup of G. Then $A = m^{-1}A$ and $A = B \oplus C$ if and only if $m^{-1}A = m^{-1}B \oplus m^{-1}C$.*

Proof: Since $m^{-1}(mA) = A + G[m] = A$ and $mA = A \cap mG$, from (3.1.)a), we obtain:

$$A = A + G[m] = m^{-1}(mA) = m^{-1}(mG \cap A) = \left[m^{-1}(mG)\right] \cap \left(m^{-1}A\right) =$$
$$= (G + mG) \cap \left(m^{-1}A\right) = G \cap \left(m^{-1}A\right) = m^{-1}A$$

Now the last statement from the enunciation comes out from the fact that if B is a direct summand of A, then B is a pure subgroup in A and in G (see [7,26.1.(a).]

Corollary 3.3.: *a) Under the conditions from (3.2) the subgroup A has D.S.I.P. if and only if $m^{-1}A$ has this property.*

b) If G is a torsion-free group and A is a pure subgroup of G (in particular a direct summand of G), then A has D.S.I.P. if and only if $m^{-1}A$ has D.S.I.P..

Remark 3.4.: In (3.2.) and (3.3.) the conditions $G[m]=0$ or A-subgroup of G with the property that $mA = mG \cap A$, are fundamental. Thus, if one of these conditions is not fulfilled, then the conclusion from (3.3.)a) may take no place, for example:

1) If $G = Z(4) \oplus Z(2)$, $A = Z(2)$-is a pure subgroup of G, $2^{-1}A = G$, $G[2] \neq 0$. It is remarked that A has D.S.I.P., according to [19,3.1.], but $2^{-1}A$ doesn't have this property.

2) If $G = Z(16) \oplus Z(4) \oplus Z(3)$ and $A \cong Z(2)$, then $2A = 0$ and $2G \cap A \neq 0$. The subgroup A has D.S.I.P. but $2^{-1}A \cong Z(2) \oplus Z(4)$ doesn't have D.S.I.P..

4. The Frattini subgroup

Now we will study the conditions for which the Frattini subgroup of an abelian group with D.S.I.P. has D.S.I.P. too. We remind that a subgroup M of A is called maximal (in A), if $M<A$ and $M \le B < A$ implies M=B, and the Frattini subgroup of A, for short F(A), is the intersection of all maximal subgroups of A.

We begin this section with the proof of the following elementary results:

Proposition 4.1.: *For any abelian group A, the following statements occur:*

a) The subgroup M is maximal in A, if and only if it is of prime index.

b) The intersection of all maximal subgroups of A of the same prime index p is pA.

c) $F(A)=\bigcap_{p\in P} pA$.

d) A is divisible if and only if $A=F(A)$.

e) If A is torsion-free and B and C are two subgroups of A, then $F(B\cap C)=F(B)\cap F(C)$

f) If C is a subgroup in F(A), there is B-subgroup in A, such that $F(B)=C$.

g) If A is torsion-free and B and C are two subgroups of A, with $t(B)\le t(C)$, *then* $t(F(B))\le t(F(C))$ *(t(G) note the type of group G).*

Proof: a) Let M be a maximal subgroup of A. If A/M is not cyclic, then there is a generator xM of A/M, with the property that $\langle x, M\rangle \ne A$; this contradicts the maximality of M. So A/M is cyclic. If A/M is infinite, then, again, there is $x \in A/M$, with the property that $\langle x, M\rangle \ne A$. It follows that A/M is finite. Let $|A:M|=n$ be the index of M in A and $A/M=\bigoplus (A/M)_p$ the direct decomposition of A/M in its p-subgroups, according to [7,8.4.]. If $k\ge 2$, then there is a p-prime number such that $M<A_p<A$, which is a contradiction of hypothesis. So M is of prime index p in A.

Conversely, we suppose that $|A:M|=p$, where p is a prime number and B is a subgroup of A with $M\le B<A$. Then B/M is a subgroup of A/M and $|B:M|=1$, in which case M=B, or $|B:M|=p$, in which case M=A. It follows that M is maximal in A.

b) Let $F_p(A)$ be the intersection of all maximal subgroups of A of the same prime index p. If M is a subgroup of A, with $|A:M|=p$, then $\forall x\in A$, $px\in M$, so $pA\subset M$. It follows that $pA\subseteq F_p(A)$ and because $F_p(A)\subseteq pA$, we obtain that $F_p(A)=pA$.

c) The statement equality results from what we proved in case b) and from the fact that $F(A)=\bigcap_p F_p(A)$.

d) If A is divisible, then pA=A, for every prime number p. It follows that A=F(A). Conversely, if A=F(A), then $A\subseteq pA$, for every prime number p. So A is p-divisible for every $p\in P$. According to [7,20.(A)], A is divisible.

e) Let be $x\in F(B)\cap F(C)$. Then for every $p\in P$, there is $b_p\in B$ and there is $c_p\in C$, such that $x=pb_p=pc_p$. From hypothesis it follows that $b_p=c_p$, and $x\in F(B\cap C)$. Hence $F(B)\cap F(C)\subseteq F(B\cap C)$, and since the converse inclusion is always valid, we obtain the statement equality.

f) Let C be a subgroup of $F(A)=\bigcap_p F_p(A)$, and $B=\{a\in A \mid \forall\, p\in P,\ pa\in C\}$. It can be easily shown that B is a subgroup of A and F(B)=C.

g) Let be $b\in B$ and n=p-height of b in B, so $n=h_p^B(b)$. Then $b\in p^n(B)$ and $b\notin p^{n+1}(B)$. Since $b\in p^{n-1}(pA)$ and $b\notin p^n(pA)$, it follows that $h_{pA}^p(b)=n-1$. It results that, if $b\in F(B)$ and $\chi_b=(k_1,\ldots,k_n,\ldots)$ is the characteristic of b in B, then the characteristic of the same b, in F(B), is $\chi_{F(B)}=(k_1-1,\ldots,k_n-1,\ldots)$. Now we consider $u\in F(B)$ and $v\in F(C)$, $t_1=t_{F(B)}(u)=(m_1,\ldots,m_n,\ldots)$ and $t_2=t_{F(C)}(v)=(l_1,\ldots,l_n,\ldots)$ the types of elements u, respectively v, represented by one characteristic of them. In accordance with what we proved above it follows that $t_1=t_B(u)=(m_1+1,\ldots,m_n+1,\ldots)$ and $t_2=t_C(v)=(l_1+1,\ldots,l_n+1,\ldots)$. According to hypothesis it follows that $t_1\le t_2$, so $m_i\le l_i$, for every $i\in I$.

In our developments we will use extensively the following result owing to Dlab ([6,Theorem 2.])

Theorem 4.2.: *If A_i, $i\in I$, is a family of abelian groups, then*

$$F\left(\bigoplus_{i\in I} A_i\right)=\bigoplus_{i\in I} F(A_i).$$

We will now present the solution of our problem.

Theorem 4.3.: *If A is a p-group with D.S.I.P., then F(A) has D.S.I.P. too.*

Proof: According to [13,Theorem 2.], we have two cases.

Case 1: If the group A is reduced idecomposable, then there is a $n\in N^*$, such that $A=Z(p^n)$. In this case $F(A)=\bigcap_{q\in P} qA=pA\cap\left(\bigcap_{q\in P\setminus\{p\}} qA\right)=Z(p^{n-1})\cap A=Z(p^{n-1})$, because for every q-prime number different from p, we have qA=A. So F(A) is a idecomposable p-group and it has D.S.I.P.

Case 2: $A=B_p\oplus C_p$, with $B_p=\bigoplus_{m_p} Z(p)$, and $C_p=0$ or $C_p=Z(p^n)$. According to (4.2.), $F(A)=F(B_p)\oplus F(C_p)$, $F(B_p)=\bigoplus_{m_p} F(Z(p))=0$, and $F(C_p)=C_p$ (see (4.1.)(d)).

Corollary 4.4.: *If A is a torsion group with D.S.I.P., then F(A) has the same property too.*

Proof: If A is as in the enunciation, then $A=\bigoplus_p A_p$, where A_p is a p-group with D.S.I.P., according to [7,8.4.] and [13,Theorem 1.]. From (4.2.) we obtain that $F(A)=\bigoplus_p F(A_p)$, where for every p, F(A_p) has D.S.I.P. (see (4.3.)). Since F(A_p), for every $p\in P$, are fully invariant in F(A), it follows that F(A) has D.S.I.P., according to [13, Lemma 1.].

Remark 4.5.: The converse of (4.3.) (so of (4.4.) too) is generally false. For example: if $A=Z(p^2)\oplus Z(p)\oplus Z(p^n)$, then $F(A)=Z(p)\oplus Z(p^n)$ has D.S.I.P., and A doesn't have this property.

Concerning the divisible groups, from (4.1.)(d) it follows the next trivial result.

Remark 4.6.: If A is a divisible group, then A has D.S.I.P. if and only if F(A) has D.S.I.P. too.

Because F(Z)=0, from (4.2.), it follows that F(A)=0, for every free group A. Therefore, according to [19,2.2.], for free groups we have a analogous result with that from (4.6.) too.

Now we will consider.

(*) $A=\bigoplus_{i\in I} A_i$,

a torsion-free gruop. If A has D.S.I.P. and it is reduced, according to [13,Theorem 6.] (or [19,5.5.]), every A_i, $i\in I$, is either completely decomposable homogeneous or it satisfies conditions from [13,Lemma 10.] (or [19,5.4.]); and if A is not reduced, then $A=\left(\bigoplus_{m_0} Q\right)\oplus\left(\bigoplus_{n} C\right)$, where C is a reduced group of rank one.

Theorem 4.7.: *If A is torsion-free decomposed according to relationship (*), and it has D.S.I.P., then F(A) has this property too.*

Proof: If A is as in the enunciation, then, as we specify above, we have three cases.

Case 1. The group A is reduced and every A_i, $i\in I$, is completely decomposable homogeneous. Then $A_i=\bigoplus_{j_i\in J_i} A_{j_i}$, where, $\{J_i\}_{i\in I}$ represents a partition of I, and for every $j_i\in J_i$, A_{j_i} are all of rank one and isomorphic. According to what we proved above, $F(A_i)=\bigoplus_{j_i\in J_i} F(A_{j_i})$ is either 0, or a reduced completely decomposable homogoneous group too. So, $F(A)=\bigoplus_{i\in I} F(A_i)$, and according to [19,5.5.], it has D.S.I.P. too.

Case 2. The group A is reduced, the set M_A of all types of elements of A contains the smallest element t_0, for which the set $I_A(t_0)=\{i\in I \mid t(A_i)=t_0\}$ is a finite subset of I, and for every i_1, $i_2\in I\setminus I_A(t_0)$, $i_1\neq i_2$, $t(A_{i_1})$ and $t(A_{i_2})$ are incomparable. From (4.1.)g) it follows that there is t_1, a the smallest element in the set $M_{F(A)}$ of all types of elements of F(A), with the property that $I_{F(A)}(t_1)=\{i\in I \mid t(F(A_i))=t_1\}$ is a finite subset of I, and for every i_1, $i_2\in I\setminus I_{F(A)}(t_1)$, $i_1\neq i_2$, $t(F(A_{i_1}))$ and $t(F(A_{i_2}))$ are incomparable. It results that together with A F(A) satisfies [19,5.5.] as well, and so it has D.S.I.P. too.

Case 3. The group A is not reduced and $A=\left(\bigoplus_{m_0} Q\right)\oplus\left(\bigoplus_{n} C\right)$, where C is a reduced of rank one group, m_0 and n being two non-null natural numbers. Then $F(A)=\left(\bigoplus_{m_0} Q\right)\oplus\left(\bigoplus_{n} F(C)\right)$, where F(C) is reduced of rank one if $C\neq Z$, or $F(C)=0$ if $C=Z$. Therefore, and in this case, F(A) has D.S.I.P.

Remark 4.8.: The converse of (4.7.) is generally false, that is, if A is a torsion-free group with property that F(A) has D.S.I.P., it does not result that A has D.S.I.P. too. For example: let be $A=\left(\bigoplus_{m_0} Q\right)\oplus\left(\bigoplus_{n} C\right)\oplus\left(\bigoplus_{m_1} Z\right)$, where m_0, n and $m_1\in N^*$, and C is torsion-free, reduced and of rank one group. Then, $F(A)=\left(\bigoplus_{m_1} Q\right)\oplus\left(\bigoplus_{r} F(C)\right)$ has D.S.I.P., but A doesn't have this property anymore.

Combining what we obtain in (4.4.), (4.7.) and [19,6.4.], we have.

Corollary 4.9.: *If A is a splitting mixed group with D.S.I.P., then F(A) has D.S.I.P. too.*

From (4.8.) it results:

Remark 4.10.: The converse of (4.9.) is generally false, that is there are abelian mixed groups, A, with property that F(A) has D.S.I.P., without A having D.S.I.P.

Let A be a p-group. In this case, as we proved in (4.3.), F(A)=pA, and so A/F(A) is an elementary p-group.

Now we will consider A as being a torsion group, and let $A=\bigoplus_p A_p$ be the direct decomposition of A in its p-subgroups, according to [7,8.4.]. In this case, $F(A)=\bigoplus_p F\left(A_p\right)=\bigoplus_p pA_p$. If $\bar{a}\in A/F(A)=\bigoplus_p A_p/(\bigoplus_p pA_p)$, then there is $k\in N^*$ and there is $a_{p_i}\in A_{p_i}$, for every $i=\overline{1,k}$, such that $\bar{a}=a_{p_1}+\ldots+a_{p_k}+\left(\bigoplus_p pA_p\right)$, and the order of a is $p_1\cdots p_k$. It follows that any element from A/F(A) has the order a square-free integer. So A/F(A)=S(A/F(A)) (S(G) being the socle of group G), and A/F(A) is a elementary group.

Synthetizing what we obtained in this paragraph, concerning A/F(A), we have the following results:

Proposition 4.11.: *Let A be an abelian group and F(A) its Frattini subgroup. In any of the following situations, A/F(A) has D.S.I.P.:*

a) A is a p-group,
b) A is a torsion group,
c) A is a free group,
d) A is a divisible group,
e) A is the direct sum between a free group and a divisible group,
f) A is the direct sum between elementary group, a divisible group and a free group.

5. The p-basic subgroups

Let A be an abelian group and p any prime number. By a p-basic subgroup B of A we mean a subgroup of A satisfying the following three conditions:

i) B is a direct sum of cyclic p-groups and infinite cyclic groups,
ii) B is p-pure in A,
iii) A/B is p-divisible.

From [7,32.3.,35.2.] it follows that every group contains p-basic subgroups, for a given prime p, all p-basic subgroups of a group are isomorphic.

If B is the p-basic subgroup of A, then:

(*) $B=B_0\oplus B_1\oplus\ldots\oplus B_n\oplus\ldots$

where: $B_0=\oplus Z$ and $B_n=\oplus Z\left(p^n\right)$ for every $n\in N^*$.

From the above it follows:

Remark 5.1.: The group A is p-basic in A, if and only if A is a direct sum of cyclic p-groups and infinite cyclic groups.

Remark 5.2.: The subgroup 0 is p-basic in A, if and only if A is p-divisible.

Remark 5.3.: From (5.2.) it follows that if A is divisible, then its p-basic subgroup is 0 and so it has D.S.I.P., indifferent if A has or has not this property.

In our developments we will use the following result very much:

Proposition 5.4.: *(Fuchs, [7]) If B_i is the p-basic subgroup of A_i, $i \in I$, then $\bigoplus_{i \in I} B_i$ is the p-basic subgroup of $\bigoplus_{i \in I} A_i$.*

Proof: If the relationship (i) is true for every B_i, $i \in I$, then it is true for $\bigoplus_{i \in I} B_i$ too. Let x be any element from $\left(\bigoplus_{i \in I} B_i\right) \cap p\left(\bigoplus_{i \in I} A_i\right) = \left(\bigoplus_{i \in I} B_i\right) \cap \left(p \bigoplus_{i \in I} A_i\right)$ (see (1.2.)a)). Then there is $n \in N^*$, such that $x = b_{i_1} + \ldots + b_{i_n} = p a_{i_1} + p a_{i_2} + \ldots + p a_{i_n}$ (1), where $a_{i_k} \in A_{i_k}$ and $b_{i_k} \in B_{i_k}$, for every $k = \overline{1,n}$. Writting the relationship (1) in the shape:

$$\left(b_{i_1} - p a_{i_1}\right) + \left(b_{i_2} - p a_{i_2}\right) + \ldots + \left(b_{i_n} - p a_{i_n}\right) = 0,$$

we obtain that: $b_{i_k} = p a_{i_k}$, for every $k = \overline{1,n}$. Then $b_{i_k} \in B_{i_k} \cap p A_{i_k} = p B_{i_k}$, $k = \overline{1,n}$, because B_i are p-pure subgroups in A_i, for every $i \in I$. It results that there is $b'_{i_k} \in B_{i_k}$, $k = \overline{1,n}$, such that $b_{i_k} = p b'_{i_k}$ and $x = p\left(b'_{i_1} + \ldots + b'_{i_n}\right) \in p\left(\bigoplus_{i \in I} B_i\right)$. Thus that $\left(\bigoplus_{i \in I} B_i\right) \cap p\left(\bigoplus_{i \in I} A_i\right) \subset p\left(\bigoplus_{i \in I} A_i\right)$. This inclusion shows that $\bigoplus_{i \in I} B_i$ is a p-pure in $\bigoplus_{i \in I} A_i$. Now let be x and y two elements from $\bigoplus_{i \in I} A_i$, $\bar{x}$ and $\bar{y}$ their cosets modulo ($\bigoplus_{i \in I} B_i$). We will consider the equation:

(2) $p\bar{x} = \bar{y}$,

in $\bigoplus_{i \in I} A_i / (\bigoplus_{i \in I} B_i)$. Then $x = x_{i_1} + \ldots + x_{i_n}$, $y = y_{j_1} + \ldots + y_{j_m}$, where m, $n \in N^*$, $x_{i_k} \in A_{i_k}$, $k = \overline{1,n}$, $y_{j_l} \in A_{j_l}$, $l = \overline{1,m}$, and the equation (2) becomes:

(3) $p\left(x_{i_1} + \ldots + x_{i_n} + \left(\bigoplus_{i \in I} B_i\right)\right) = y_{j_1} + \ldots + y_{j_m} + \left(\bigoplus_{i \in I} B_i\right)$,

or equivalent:

(4) $p x_{i_k} = y_{i_k} + b_{i_k}$, $k = \overline{1,n}$,

and $b_{i_k} \in B_{i_k}$. Thus we could write the equations (4):

(5) $p x_{i_k} + B_{i_k} = y_{i_k} + B_{i_k}$, $k = \overline{1,n}$.

Since A_i / B_i are p-divisible groups, $\forall i \in I$, there is $a_{i_k} \in A_{i_k}$, $k = \overline{1,n}$, such that:

(6) $p\left(a_{i_k} + B_{i_k}\right) = y_{i_k} + B_{i_k}$,

that is the equations (5) have solutions in A_{i_k} / B_{i_k}. By summing all the relationships (6) we obtain:

(7) $p\left(a_{i_1}+\dots+a_{i_k}+\left(\bigoplus B_i\right)\right)=y+\left(\bigoplus_{i\in I} B_i\right)$.

It follows that the equation (2) has the solution $\bar{a}=a+\left(\bigoplus_{i\in I} B_i\right)$, where $a=a_{i_1}+\dots+a_{i_k}$, so $\bigoplus_{i\in I} A_i/(\bigoplus_{i\in I} B_i)$ is p-divisible. Thus we proved that the relationships (i)-(iii), from the definition of p-basic subgroups are also true for $\bigoplus_{i\in I} B_i$, which is a subgroup of $\bigoplus_{i\in I} A_i$. Therefore $\bigoplus_{i\in I} B_i$ is the p-basic subgroup of $\bigoplus_{i\in I} A_i$.

We will note from this point on with B_A the p-basic subgroup of A.

Proposition 5.5.: *If A is a p-group with D.S.I.P., then B_A has D.S.I.P. too.*

Proof: If $A=Z(p^n)$, $n\in N^*$, then $A=B_A$, according to (5.1.). If $A=\left(\bigoplus_{m_r} Z_r\right)\oplus C_r$, then $B_A=\bigoplus_{m_r} Z_r$, because $B\left(\bigoplus_{m_r} Z(p)\right)=\bigoplus_{m_r} Z(p)$ and $B_{C_r}=0$ (see (5.1.), (5.2.) and (5.4.)).

Corollary 5.6.: *If A is a torsion group with D.S.I.P., then B_A has D.S.I.P. too.*

Proof: Let $A=\bigoplus_p A_p$ be a torsion group with D.S.I.P. Then according to (5.4.) and (5.5.), $B_A=\bigoplus_p B_{A_p}$, where every B_{A_p} is a p-group with D.S.I.P. Now [13,Lemma 1.] completes the demonstration.

Theorem 5.7.: *If $C\neq Z$ is a reduced torsion-free of rank one group, then $B_C=0$.*

Proof: If C is as in the enunciation, then C is a subgroup of **Q**. Since the rank of B_C is at the most equal with one and B_C has the form from (*), it follows that either $B_C=0$ or $B_C=$**Z**. If $B_C=$**Z**, then **Z** is p-pure in C, that is $pZ=Z\cap pC$ (**). If C contains at least one fraction with the denominator p, then the equality (**) is not true and **Z** is not p-pure in C. Now we suppose that C does not contain any rational fraction with the denominator p and that C/**Z** is p-divisible. Then, for every y from C, the equation p(x+**Z**)=y+**Z** has a solution x+**Z**∈C/**Z**, that is $\forall y\in C, \exists x\in C$, such that $px-y=k\in Z$; where it follows that $x=\frac{y+k}{p}\in C$. If y∉**Z**, then x is a rational fraction with the denominator p, from C-which is in contradiction with our supposition and if $Z\not\subset C$, then $B_C=0$. Therefore **Z** cannot be the p-basic subgroup of C.

Corollary 5.8.: *If A is a torsion-free group with D.S.I.P., then B_A has D.S.I.P. too.*

Proof: If A is an unreduced torsion-free group with D.S.I.P., then $A=\left(\bigoplus_{m} Q\right)\oplus\left(\bigoplus_{n} C\right)$ (see [19,5.16.]), where m and n are non-null natural numbers, and C is a reduced of rank one group. From (5.3.), (5.2.) and (5.7.) it follows that $B_A=0$, if C≠**Z**, and if C=**Z**, then $B_A=\bigoplus_n Z$, and B_A has D.S.I.P., according [19,2.2.]. If A is a reduced group, then A is either a homogeneous group, completely decomposable, in which case $B_A=0$, or if it satisfies conditions from [19, (5.4.)], then $B_A=\oplus Z$ or $B_A=0$.

Corollary 5.9.: *If A splitting mixed group with D.S.I.P., then B_A has D.S.I.P. too.*

Proof: If A is as in the enunciation, then

$$A=\left(\bigoplus_{p\in P} A_p\right)\oplus\left(\bigoplus_{m_0} Q\right)\oplus\left(\bigoplus_{n} C\right),$$

where A_p are reduced p-groups with D.S.I.P., for every $p\in P$, and m_0, n and C have the same meaning like (5.8.). Applying (5.4.), (5.6.), (5.7.) and (5.2.), we obtain the statement.

The remarks impose themselves here.

Remark 5.10.: The converse of (5.3.) is generally false. Indeed, if $A=Z(p^n)\oplus Z(p^m)$, then $B_A=0$ has in trivial way D.S.I.P., and A doesn't have D.S.I.P. (see [13,Lemma 2.]).

Remark 5.11.: From (5.10.) and (5.2.) it follows that the converse of (5.5.) is generally false. So there is an abelian p-group with the property that B_A has D.S.I.P., without A having D.S.I.P..

Remark 5.12.: From (5.7.) it follows that the converse of (5.8.) is generally false, that is: if A is a torsion-free group and B_A has D.S.I.P., it does not result that A has this property too.

Remark 5.13.: From (5.10.) (or (5.12.)) it follows that the converse of (5.9.) is generally false.

In the end we present some classes of abelian groups, for which the quatient group A/B_A has D.S.I.P..

Theorem 5.14.: *Let A be an abelian group, p a any prime number and B_A the p-basic subgroup of A. In any of the following situations, the group A/B_A has D.S.I.P.:*

a) A is a p-group with D.S.I.P.,

b) A is a torsion group with D.S.I.P.,

c) A is a free group,

d) A is a divisible group with D.S.I.P.,

e) $A=\bigoplus_{i\in I} A_i$ is a torsion-free group with D.S.I.P., which does not satisfy [19.5.4.],

f) A is a group of the form:

$$\left(\bigoplus_{p} A_p\right)\oplus D\oplus\left(\bigoplus_{m_0} Z\right)\oplus\left[\bigoplus_{i\in I}\left(\bigoplus_{m_i} C_i\right)\right],$$

where: -for every $p\in P$, A_p is a p-group with D.S.I.P.,

-D is a divisible group,

-m_0 and m_i, $i\in I$, can be any natural numbers,

-for every $i\in I$, C_i is a torsion-free completely decomposable homogeneous and reduced group.

Proof: a) If $A=Z(p^n)$, with $n\in N^*$, then $B_A=A$, and if $A=\left(\bigoplus_{m_p} Z(p)\right)\oplus C_p$, with $m_p\in N$ and $C_p=0$ or $C_p=Z(p^\infty)$, then $B_A=\bigoplus_{m_p} Z(p)$. Hence, in this case, either $A/B_A=0$ or $A/B_A=Z(p^\infty)$.

b) Let A be a torsion group with D.S.I.P. and let $A=\left(\bigoplus_{p\in P_0} Z(p^{n_p})\right)\oplus\left[\bigoplus_{p\in P\setminus P_0}\left(\bigoplus_{m_p} Z(p)\right)\right]\oplus\left(\bigoplus_{p\in P\setminus P_0} C_p\right)$ be a direct decomposition of A, where $P_0\subset P$, $n_p\in N^*$, $m_p\in N$ and C_p have the same meaning as above. In this case $B_A=\left(\bigoplus_{p\in P_0} Z(p^{n_p})\right)\oplus\left[\bigoplus_{p\in P\setminus P_0}\left(\bigoplus_{m_p} Z(p)\right)\right]$, and $A/B_A=\bigoplus_{p\in P\setminus P_0} C_p$. Now [19,4.4.] completes the demonstration.

c) If A is a free group, then $B_A=A$ and $A/B_A=0$.

d) If A is a divisible group, then A is p-divisible and, according to (5.2.), $B_A=0$. Thus, in this case $A/B_A\cong A$.

e) At first we suppose that the gruop A is reduced. In this case, according to [19,5.5.], for every $i\in I$, A_i is homogeneous group, completely decomposable. Let be $i\in I$ and $A_i=\bigoplus_{j_i\in J_i} A_{j_i}$, where $\{J_i\}_{i\in I}$ is a partition of I, and for every $j_i\in J_i$, A_{j_i} are of rank one and isomorphic. From (5.4.) and (5.7.) it follows that $B_{A_i}=\bigoplus_{j_i\in J_i} B_{A_{j_i}}=\bigoplus_{j_i\in J_i} \mathbf{Z}$, if $\forall j_i\in J_i$, $A_{j_i}=\mathbf{Z}$, or $B_{A_i}=0$, if $\forall j_i\in J_i$, $A_{j_i}\neq\mathbf{Z}$. So B_A is either 0 or a free group. Let be $I_0=\{i\in I \mid A_i$ is a free group$\}$. It results that, in this case, $B_A=\bigoplus_{i\in I_0} B_{A_i}=\bigoplus_{i\in I_0} A_i$, and $A/B_A\cong\bigoplus_{i\in I\setminus I_0} A_i$. Now, we suppose that A is not reduced. Then, $A=\left(\bigoplus_{m_0} Q\right)\oplus\left(\bigoplus_{n} C\right)$, where C is a reduced group of rank one, m_0 and n being two not-null natural numbers. It follows that either $B_A=0$, if $C\neq\mathbf{Z}$, or $B_A=\bigoplus_{n} C$, if $C=\mathbf{Z}$. So, in this case either A/B_A is isomorphic with A, or it is a torsion-free divisible group, and according to [19,4.1.], A/B_A has D.S.I.P..

f) The statement results from (5.4.), (5.5.), (5.1.), (5.2.) and (5.7.)

From [19,6.4.] and (5.14.)f) we obtain:

Corollary 5.15.: *If A is a splitting mixed group with D.S.I.P., and B_A is the p-basic subgroup of A, where p is a prime number, then A/B_A has D.S.I.P..*

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