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ABOUT THE REMAINDER OF AN INTERPOLATION FORMULA OVER TRIANGLES

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Abstract. The goal of the paper is to present a method to obtain some interpolants which match a function and certain of its derivatives on the boundary of a triangle and to compute the remainders of those interpolants.

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Key words: blending interpolant, boolean sum operator.

1. Preliminaries

Beginning with the paper by Barnhill, Birkhoff and Gordon [1], the interpolation problem to boundary data on a triangle was largely studied. Considering the standard triangle $T_h = \{(x,y) \in R^2 | x \geq 0, y \geq 0, x + y \leq h\}$ with the vertices $V_1 = (h,0), V_2 = (0,h), V_3 = (0,0)$ and with the opposite sides denoted by E_1, E_2, E_3 (fig.1) in the paper [1] there are constructed some interpolants which match a *given* function $f: T_h \to R$ on the sides of T_h.

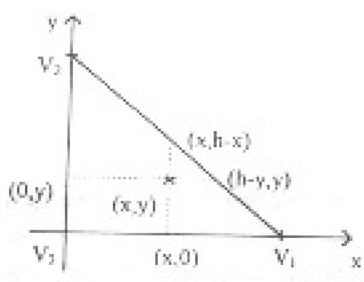

Fig. 1

Let be $(x,y) \in Int T_h$ and let be L_1^x the linear interpolation operator along the parallel to the side E_2 and L_1^y the linear interpolation operator along the parallel to the side E_1. From [1] the interpolation operators have the expressions

$$(1.1)\qquad L_1^x(f)(x,y) = \frac{h-x-y}{h-y} f(0,y) + \frac{x}{h-y} f(h-y,y)$$

$$L_1^y(f)(x,y) = \frac{h-x-y}{h-x} f(x,0) + \frac{y}{h-x} f(x,h-x)$$

Important contributions to the development of the theory of interpolation over triangle are due to the Romanian mathematicians Gh. Coman, I. Gânscă and L. Ţâmbulea [4], [5], [6], [7].

Let H_2^y be the Hermite interpolation operator along the parallel to the side E_1 which matches the function f and its first partial derivative with respect to y at the point (x,0) and also the function f at (x,h-x) and the Hermite interpolation H_2^x the Hermite interpolation operator along the parallel to the side E_2 which matches the function f and its first partial derivative with respect to x at the point (0,y) and also the function f at the point (h-y,y).

In [3] one has established the following results.

Lemma 1.1. The equalities

$$(1.2.1)\qquad H_2^y(f)(x,y) = \frac{y(h-x-y)}{h-x} f^{(0,1)}(x,0) + \frac{y^2}{(h-x)^2} f(x,h-x) + \frac{(h-x-y)(h-x+y)}{(h-x)^2} f(x,0)$$

$$(1.2.2)\qquad H_2^x(f)(x,y) = \frac{x(h-x-y)}{h-y} f^{(1,0)}(0,y) + \frac{x^2}{(h-y)^2} f(h-y,y) + \frac{(h-x-y)(h-x+y)}{(h-y)^2} f(0,y)$$

holds for any $x \in [0,h)$, $y \in [0,h)$.

Remark 1.2. $H_2^y(f)$ interpolates the function f and its first partial derivative with respect to y on the side E_1 and also the function f on the side E_3 and $H_2^x(f)$ interpolates the function f and its first partial derivative with respect to x on the side E_2 and also the function f on the side E_3.

Lemma 1.3. The blending interpolants $(L_1^x \oplus H_2^y)(f)$ and $(L_1^y \oplus H_2^x)(f)$ have the following expressions

(1.3.1) $$(L_1^x \oplus H_2^y)(f)(x,y) = \frac{y(h-x-y)}{h(h-x)}\left\{hf^{(0,1)}(x,0)+(x-h)f^{(0,1)}(0,0)\right\}+$$

$$+\frac{1}{(h-x)^2}\left\{y^2 f(x,h-x)+(h-x-y)(h-x+y)f(x,0)\right\}+\frac{h-x-y}{h-y}f(0,y)-$$

$$-\frac{h-x-y}{h^2(h-y)}\left\{y^2 f(0,h)+(h^2-y^2)f(0,0)\right\}$$

(1.3.2) $$(L_1^y \oplus H_2^x)(f)(x,y) = \frac{x(h-x-y)}{h(h-y)}\left\{hf^{(1,0)}(0,y)+(y-h)f^{(1,0)}(0,0)\right\}+$$

$$+\frac{1}{(h-y)^2}\left\{x^2 f(h-y,y)+(h-x-y)(h-x+y)f(0,y)\right\}+\frac{h-x-y}{h-x}f(x,0)-$$

$$+\frac{h-x-y}{h^2(h-x)}\left\{(h^2-x^2)f(0,0)-x^2 f(h,0)\right\}$$

Theorem 1.1. The operators $(L_1^x \oplus H_2^y)(f)$ and $(L_1^y \oplus H_2^x)(f)$ have the properties

(1.4) $$(L_1^x \oplus H_2^y)(f) = f \ on\ \partial T_h$$

$$(L_1^x \oplus H_2^y)^{(0,1)}(f) = f^{(0,1)} \ on\ E_1$$

$$(L_1^y \oplus H_2^x)(f) = f \ on\ \partial T_h$$

$$(L_1^y \oplus H_2^x)^{(1,0)}(f) = f^{(1,0)} \ on\ E_2$$

2. Main results.

Using the boolean sum operators we can consider now the following approximation formula

$$f = (L_1^x \oplus H_2^y)(f) + R_{12}^{xy} f$$

$$f = (L_1^y \oplus H_2^x)(f) + R_{12}^{yx} f$$

where R_{12}^{xy} and R_{12}^{yx} are the corresponding remainder terms.

Theorem 2.1. If $f \in B_{12}(0,0)$ then the remainder term has the expression

$$(R_{12}^{xy} f)(x,y) = \int_0^h K_{20}(x,y,s) f^{(2,0)}(s,0)ds + \int_0^h K_{21}(x,y,s) f^{(2,1)}(s,0)ds +$$

$$+\int_0^h K_{02}(x,y,t) f^{(0,2)}(0,t)dt + \iint_{T_h} K_{12}(x,y,s,t) f^{(1,2)}(s,t)dsdt$$

where

$$K_{30}(x,y,s)=\frac{(x-s)_+^2}{2},\quad K_{21}(x,y,s)=y\cdot(x-s)_+$$

$$K_{02}(x,y,t)=\frac{y^2}{(h-x)^2}\cdot\frac{(h-x-t)_+^2}{2}+\frac{h-x-y}{h-y}\cdot\frac{(y-t)_+^2}{2}-\frac{y^2(h-x-y)(h-t)^2}{2\cdot h^2(h-y)}$$

$$K_{12}(x,y,s,t)=\frac{y}{h-x}(x-s)_+^0[h-x-y+\frac{y}{h-x}(h-x-t)_+]$$

Proof. Taking into account that $R_{12}^{T}f=f,(\forall)f\in P_2^2$, the proof follows by the Sard kernel theorem in triangles [4], with

$$K_{30}(x,y,s)=(I_1^x\oplus H_2^y)\left[\frac{(x-s)_+^2}{2}\right],$$

$$K_{21}(x,y,s)=(I_1^x\oplus H_2^y)[(x-s)_+\cdot y]$$

$$K_{02}(x,y,t)=(I_1^x\oplus H_2^y)\left[\frac{(y-t)_+^2}{2}\right],$$

$$K_{12}(x,y,s,t)=(I_1^x\oplus H_2^y)[(x-s)_+^0\cdot(y-t)_+]$$

Theorem 2.2. If $f\in B_{21}(0,0)$ then

$$(R_{12}^{T}f)(x,y)=\int_0^h K_{30}(x,y,s)f^{(3,0)}(s,0)ds+\int_0^h K_{12}(x,y,s)f^{(1,2)}(s,0)ds+$$

$$+\int_0^h K_{02}(x,y,t)f^{(0,2)}(0,t)dt+\iint_{T_h}K_{21}(x,y,s,t)f^{(2,1)}(s,t)dsdt$$

where

$$K_{03}(x,y,s)=\frac{(y-t)_+^2}{2}$$

$$K_{12}(x,y,s)=x\cdot(y-t)$$

$$K_{01}(x,y,t)=\frac{x^2}{(h-y)^2}\cdot\frac{(h-y-s)_+^2}{2}+\frac{h-x-y}{h-x}\cdot\frac{(x-s)_+}{2}-\frac{x^2(h-x-y)(h-s)^2}{2h^2(h-x)}$$

$$K_{21}(x,y,s,t)=\frac{x}{h-y}(y-t)_+^0[h-x-y+\frac{x}{h-y}(h-y-s)_+]$$

Proof. Taking into account that $R_{12}^{m} f = f, (\forall) f \in P_2^2$, the proof follows by the Sard kernel theorem in triangles [4], with

$$K_{20}(x,y,s) = (L_1^y \oplus H_2^x)\left[\frac{(x-s)_+^2}{2}\right],$$

$$K_{11}(x,y,t) = (L_1^y \oplus H_2^x)\left[(y-t)_+ \cdot x\right]$$

$$K_{02}(x,y,t) = (L_1^y \oplus H_2^x)\left[\frac{(y-t)_+^2}{2}\right],$$

$$K_{12}(x,y,s,t) = (L_1^y \oplus H_2^x)\left[(x-s)_+ \cdot (y-t)_+^0\right]$$

Remark 2.3. The multiplicity of the knots (x,0), (x,h-x) respectively (0,y), (h-y,y) can be inverted.

Remark 2.4. Considering the boolean sum operator $H_2^x \oplus H_2^y$ one obtains the blending function interpolant $(H_2^x \oplus H_2^y)(f)$. This function interpolates f on ∂T_h and its first partial derivatives $f^{(0,1)}$ *and* $f^{(1,0)}$ on E_1 and respectively on E_2.

Remark 2.5. The interpolation procedures which was presented above have many applications in computer aided geometry(see [6], [7]).

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