Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol.XIII (1997),45-50

EQUIVALENT DOMAIN INTEGRAL EVALUATION IN FRACTURE PROBLEMS

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SUMMARY

For mixed - mode fracture problems in three - dimensional solids, the general formulation of the equivalent domain integral (EDI) method is presented. The total J - integral is split into J_I, J_{II}, and J_{III} for the three modes of deformation. The method can be used as a post - processing subroutine in the SAPLI linear elastic finite element program.

1. INTRODUCTION

In practical cases, cracks in structures have arbitrary three - dimensional shapes and the crack faces may be non - planar. The actual trend is to calculate fracture mechanics parameters as a three - dimensional problem. Besides, for practical mixed - mode fracture problems the decomposition into the three pure modes is not an easy task. In finite element analyses different numerical techniques have been developed. Two of them: the virtual crack extension (VCE) method and the J - integral method can be applied to both linear and non-linear problems. When there is not available an analytical solution, it is generally recognised that energetic methods of fracture parameters calculation have a higher rate of convergence to the hypothetical exact solution. This is why the VCE method, [1 - 4], is extensively used. The J - integral method is very attractive, particularly for non-linear material problems. For two - dimensional (2 - D) problems, several crack tip integrals, [5 -8], were developed and calculated as the sum of a remote line integral and an area integral around the crack tip. For three - dimensional (3 - D) problems, the J - integral is the sum of a remote surface integral and a volume integral around the crack front. The evaluation of surface integrals which include singular terms is unfriendly in finite element (FE) analysis.

An important step was taken when de Lorenzi [9,10] introduced the S - function concept to define the virtual crack extension in 3 - D cracked solids. Afterwards, the J - integral formulation has been modified into a domain integral form. The surface integrals for 3 - D problems can be transformed into integrals over a domain or volume, and the method was named the *equivalent domain integral* (EDI) method, [11 - 14]. This method is revised, [15], and applied to mixed - mode fracture problems. The direct and decomposition methods are used to separate the modes. It is also presented a general formulation of the EDI method for the calculation of the J - integral under mixed - mode loading conditions.

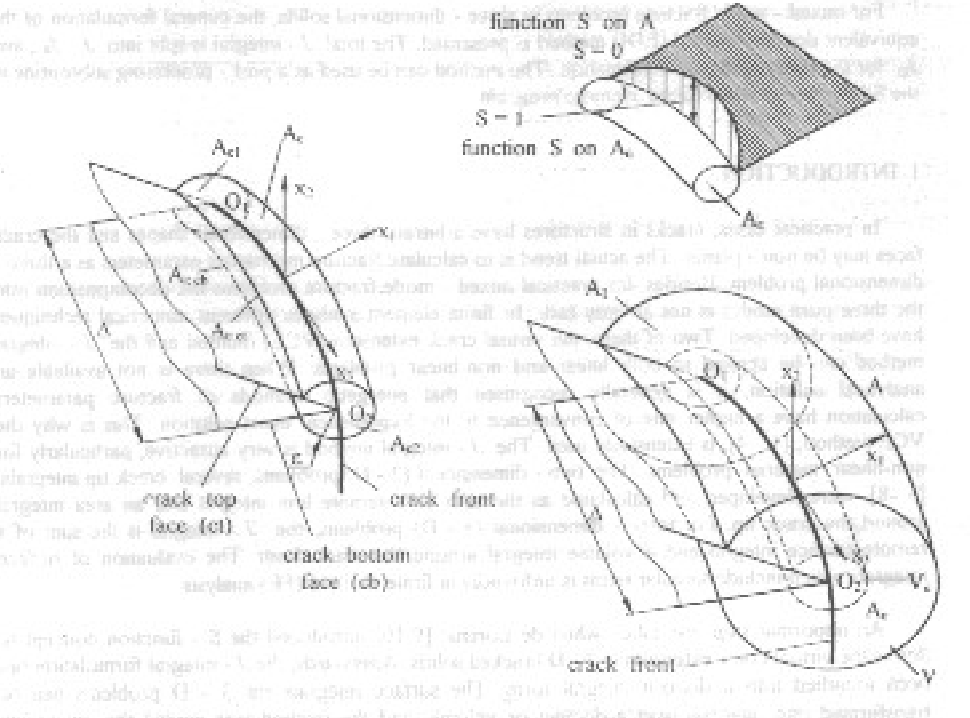
2. THE EQUIVALENT DOMAIN INTEGRAL [15]

In 3 - D crack problems, the crack front forms a line singularity and the strength of the singularity may be varying all along the crack front. At a point of the crack front the path independence is valid only over a small region, due to interactive singular fields of neighbouring points on the crack front. A small tube of radius ε is considered around a segment of crack front of length Δ as in figure 1a;

at the limit, Λ and ε / Λ tends to zero. The crack front integral, over the surface A_ε is defined as:

$$\int_\Lambda J_{x_k} dx_3 = \lim_{\substack{\varepsilon/\Lambda \to 0 \\ \Lambda \to 0}} \iint_{A_\varepsilon} \left[W n_k - \sigma_{ij} \frac{\partial u_i}{\partial x_k} n_j \right] dA \qquad (1)$$

In equation (1), W is the stress - work density, σ_{ij} is the stress tensor, u_i is the displacement vector, and n_k is the k-th directional component of the unit normal vector on the closed surface A_ε. The indices i and j take the values 1, 2, and 3, and k takes the values 1 and 2. The local value of J_{x_k} is the total energy flux leaving the closed surface A_ε per unit crack front length in the k-th direction. Axes x_1 and x_3 are in the crack plane and are normal, respectively tangential to the crack front, while x_2 is normal to the crack plane.



a) crack front and enclosed surface A_ε b) domain around the crack front

Fig. 1 Nomenclature and the domain description

The complete surface integral is:

$$\int_\Lambda J_{x_k} dx_3 = \int_{A_\varepsilon} QdA + \int_{A_1 + A_2} QdA + \int_{A_{ct} + A_{cb}} QdA \qquad (2)$$

where

$$Q = \left[W n_k - \sigma_{ij} \frac{\partial u_i}{\partial x_k} n_j \right] \qquad (3)$$

$$W = \int_0^{\varepsilon} \sigma_{ij} d\varepsilon_{ij} \quad (4)$$

In equation (2), A_{c_1} and A_{c_2} are cross sectional areas of the tube at O_1 and O_2. The subscripts ct and cb represent top and bottom crack surfaces. The total strain tensor ε_{ij} includes elastic, plastic, and thermal components. For linear elastic problems, $W = (\sigma_{ij} \cdot \varepsilon_{ij})/2$. To calculate stresses from strains, the appropriate constitutive relations (isotropic or anisotropic) must be used. Therefore, equation (2) may be applied to general problems.

In a 3 - D cracked solid, the three fracture modes are: opening (mode I), shearing (mode II), and tearing (mode III). The corresponding three modes of the J - integral are J_I, J_{II}, and J_{III}. In equations (1) or (2), J_{c_1} and J_{c_2} represent the total J - integral $[J_I + J_{II} + J_{III}]$ and the product integral $\left[-2\sqrt{J_I \cdot J_{II}}\right]$, respectively [8,11,12]. Since the meaning of J_{c_3} integral is not clear for crack problems, it is not defined by the previous equations. The mode III integral is separately defined as, [12].

$$\int_{\Delta} J_{III}\, dx_3 = \lim_{\substack{\varepsilon \to 0 \\ \Delta \to 0}} \left[\int_{A_\varepsilon} Q_3\, dA + \int_{A_{c_1}+A_{c_2}} Q_3\, dA + \int_{A_{ct}+A_{cb}} Q_3\, dA \right] \quad (5)$$

where

$$Q_j = W^{III} n_j - \sigma_{ij} \frac{\partial u_i}{\partial x_j} n_i \quad (6)$$

$$W^{III} = \int_0^{\varepsilon} \sigma_{3j}\, d\varepsilon_{3j} \quad (7)$$

The indices i and j take values 1, 2, and 3. Equations 2 and 5 could introduce errors due to numerical integration of singular terms. The evaluation of the surface integral is transformed to an equivalent domain integral.

If a virtual crack extension of a crack front is made, an arbitrary but continuous function $S(x_1, x_2, x_3)$ is introduced [11, 12, 15] that has the property $S(x_1, x_2, x_3) = 0$ on the surface A and $S(x_1, x_2, x_3) = S(x_3)$ on the surface A_ε. Using the S - function it results:

$$\int_{\Delta} J_{c_k} S\, dx_3 = \int_{A_\varepsilon} Q S\, dA + \int_{A_{c_1}+A_{c_2}} Q S\, dA + \int_{A_{ct}+A_{cb}} Q S\, dA - \int_{A} Q S\, dA \quad (8)$$

$$\int_{\Delta} J_{III} S\, dx_3 = \int_{A_\varepsilon} Q_3 S\, dA + \int_{A_{c_1}+A_{c_2}} Q_3 S\, dA + \int_{A_{ct}+A_{cb}} Q_3 S\, dA - \int_{A} Q_3 S\, dA \quad (9)$$

As the considered length Δ of the crack front is small, ($\lim \Delta \to 0$), J_{c_k} is assumed to be constant over Δ and equations (8) and (9) may be rewritten:

$$J_{c_k} \cdot f = \int_{A_\varepsilon} Q S\, dA + \int_{A_{c_1}+A_{c_2}} Q S\, dA + \int_{A_{ct}+A_{cb}} Q S\, dA - \int_{A} Q S\, dA \quad (10)$$

$$J_{III} \cdot f = \int_{A_\varepsilon} Q_3 S\, dA + \int_{A_{c_1}+A_{c_2}} Q_3 S\, dA + \int_{A_{ct}+A_{cb}} Q_3 S\, dA - \int_{A} Q_3 S\, dA \quad (11)$$

where $$f = \int_{O_1}^{O_2} S(x_3)\,dx_3 \,, \qquad (12)$$

The parameter f is equivalent to the new crack surface area created by translating the crack front by $S(x_3)$ in the x_1 - direction. By selecting the S - function such that it has zero values at two end surfaces (O_1 and O_2) of the tubes A_t and A and nonzero between these two end faces, the second surface integrals in equations (10) and (11) become identically zero.

Hence, J_{x_1} and J_{II} are expressed as the sum of domain and crack surface integrals as follows.

$$J_{x_1} \cdot f = \left(J_{x_1} \cdot f\right)_{domain} + \left(J_{x_1} \cdot f\right)_{crack\ faces} \qquad (13)$$

$$J_{II} \cdot f = \left(J_{II} \cdot f\right)_{domain} + \left(J_{II} \cdot f\right)_{crack\ faces} \qquad (14)$$

Using Green's divergence theorem, the closed surface integrals are transformed, after some algebraic calculations, into domain integrals.

$$\left(J_{x_1} \cdot f\right)_{domain} = -\int QSdA = -\int\left[Wn_1 - \sigma_{ij}\frac{\partial u_i}{\partial x_1}n_j\right]SdA = -\int_{(V-V_t)}\left[\frac{\partial (WS)}{\partial x_1} - \frac{\partial}{\partial x_j}\left(\sigma_{ij}\frac{\partial u_i}{\partial x_1}S\right)\right]dV, \quad (15)$$

hence

$$\left(J_{x_1} \cdot f\right)_{domain} = -\int_{(V-V_t)}\left[W\frac{\partial S}{\partial x_1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1}\frac{\partial S}{\partial x_j}\right]dV - \int_{(V-V_t)}\left[\frac{\partial W}{\partial x_1} - \sigma_{ij}\frac{\partial \varepsilon_{ij}}{\partial x_1}\right]SdV \qquad (16)$$

$$\left(J_{II} \cdot f\right)_{domain} = -\int_{(V-V_t)}\left[W^{II}\frac{\partial S}{\partial x_1} - \sigma_{ij}\frac{\partial u_i}{\partial x_1}\frac{\partial S}{\partial x_j}\right]dV - \int_{(V-V_t)}\left[\frac{\partial W^{II}}{\partial x_1} - \sigma_{ij}\frac{\partial}{\partial x_1}\left(\frac{\partial u_i}{\partial x_j}\right)\right]SdV \qquad (17)$$

In the previous relations, the equations of equilibrium $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$ and the small deformation strain - displacement relationships $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ were used. The terms in brackets in the second integral in equations (16) and (17) are equal to zero for a linear elastic material. These terms are not zero in elastic - plastic and thermal problems.

The crack - faces integrals are:

$$\left(J_{x_1} \cdot f\right)_{crack\ faces} = \int_{(A-A_t)_t} QSdA + \int_{(A_t-A)_b} QSdA + \int_{A_{ct}+A_{cb}} QSdA \qquad (18)$$

and

$$\left(J_{II} \cdot f\right)_{crack\ faces} = \int_{(A-A_t)_t} Q_2SdA + \int_{(A_t-A)_b} Q_2SdA + \int_{A_{ct}+A_{cb}} Q_2SdA \qquad (19)$$

When Q and Q_2 are zero on the crack faces, the integrals in equations (18) and (19) vanish. On the crack faces, n_1 and n_3 are always zero, while $n_2 = -1$ on the top face (ct) and $n_2 = 1$ on the bottom face (cb). For traction free crack faces the terms $\left(J_{x_1}\right)_{crack\ faces}$ and $\left(J_{II}\right)_{crack\ faces}$ vanish. In contrast, the term $\left(J_{k_2}\right)_{crack\ faces}$ is no longer zero. The $\left(J_{k_2}\right)_{crack\ faces}$ is zero only for pure mode I crack problems or for a singular stress field alone.

In mixed - mode problems, in order to separate the three modes of deformation the *direct* and *decomposition* methods may be used. They are described in detail in [15].

The S - function is an arbitrary but continuous function with zero value on the surface A and at the ends of the tube (A_{c_1} and A_{c_2}) and nonzero value (varying between zero and one) on the surface A_ε (see figure 1). On the tube face A_ε, the S - function is a function of only x_3 and has a value of one at the location where the J - integral is required. The S - functions are defined by specifying the values of S at the nodes of an eight - noded or 20 - noded isoparametric element and using the element shape functions. The values of J - equation (12) - depend only on the variation of the S - function in the x_3 direction.

3. CONCLUDING COMMENTS

The equivalent domain integral method is a powerful procedure in analysing 3 - D crack problems. We started from the idea that this method can be used as a post - processing subroutine in a general finite element program. Therefore, for the beginning, a linear elastic program was chosen as being the SAPLI program implemented on PCs. It contains the 20 - noded isoparametric element that has to be the starting point for the 3 - D mixed - mode J - integral evaluation.

The whole numerical post - processing procedure was understood and connected to the main program. The applications of this method will be presented elsewhere.

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Received 20.10.1997

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