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ON THE GENERAL LINEAR INTEGRATION METHODS

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Abstract. For the numerical solution of the initial value problem (1.1) - (1.2) a family of DIMSIM methods (Diagonally Implicit Multi-Stage Integration Methods) of order p=3, are derived.

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1. Introduction

We consider the initial value problem

$$(1.1)\qquad y' = f(x, y(x)),$$

$$(1.2)\qquad y(x_0) = y_0,$$

with $f: [a,b] \times \mathbb{R}^m \to \mathbb{R}^m$, $x_0 = a$, $y_0 \in \mathbb{R}^m$.

For the numerical solution of this problem, the general linear methods with s stages are defined by

$$(1.3)\qquad Y_i^{[n+1]} = h\sum_{j=1}^{s} a_{ij} f\left(x_n + c_j h, Y_j^{[n+1]}\right) + \sum_{j=1}^{r} u_{ij} y_j^{[n]},\ i = 1,2,\dots,s$$

$$(1.4)\qquad y_i^{[n+1]} = h\sum_{j=1}^{s} b_{ij} f\left(x_n + c_j h, Y_j^{[n+1]}\right) + \sum_{j=1}^{r} v_{ij} y_j^{[n]},\ i = 1,2,\dots,r$$

for $n=0,1,2,\ldots,N-1$, $h=\dfrac{b-a}{N}$, $x_n=x_0+nh$, $n=0,1,2,\ldots,N$.

The methods given by (1.3)-(1.4) are characterized by the real matrices

$$A=(a_{ij}),\quad B=(b_{ij}),\quad U=(u_{ij}),\quad V=(v_{ij})$$

and they include as particular cases many known numerical methods: linear multistep methods, Runge-Kutta methods, predictor-corrector methods, pseudo Runge-Kutta methods,etc.
Methods of the form (1.3)-(1.4) are investigated in [2],[3],[4],[5],[6]

The values $Y_i^{[n+1]}$, $i=1,2,\ldots,s$, are called the **internal stages** and they provide approximations of the solution of the problem (1.1) - (1.2) at the points x_n+c_ih, $i=1,2,\ldots,s$. Also, the values $y_i^{[n+1]}$, $i=1,2,\ldots,s$, are called the **external stages** (or external approximations) and they provide approximations of the solution of the problem (1.1)-(1.2) at the end of the $n-th$ step for the input to the next step, that is at the point $x_{n+1}=x_n+h$.

Similarly, as for the Runge-Kutta methods, the matrix A determines the implementation costs of the general linear method (1.3)-(1.4), and if this matrix is strictly lower triangular, then we have the **explicit** methods and in opposite case the method (1.3)-(1.4) are **implicit.**

One special case of implicit methods investigated by **J.C.Butcher**,[3], and called, " Diagonally Implicit Multi-Stage Integration Methods " (DIMSIM), are characterized by a matrix A with lower triangular form having a constant value λ on the diagonal. This class of methods can be devided into four types, depending on the structure of the matrix A.

The matrix V determines the stability of the method (1.3)-(1.4).
The general linear method (1.3)-(1.4) is **stable** (or zero-stable) if the matrix V is power bounded, i.e

$$\sup_n \| V^n \| < \infty.$$

Using Kroneker product $\otimes$ of two matrices, the general linear methods (1.3)-(1.4) can be rewrite in the form

(1.5) $$Y^{[n+1]}=h(A\otimes I)\,F(x_{n+c},Y^{[n+1]})+(U\otimes I)y^{[n]},$$

(1.6) $$y^{[n+1]}=h(B\otimes I)\,F(x_{n+c},Y^{[n+1]})+(V\otimes I)y^{[n]},$$

where

$$Y^{[n+1]} := [(Y_1^{[n+1]})^T, (Y_2^{[n+1]})^T, \ldots, (Y_s^{[n+1]})^T]^T,$$

$$y^{[n+1]} := [(y_1^{[n+1]})^T, (y_2^{[n+1]})^T, \ldots, (y_r^{[n+1]})^T]^T,$$

$$F(x_{n+c}, Y^{[n+1]}) := [f(x_n + c_1 h, Y_1^{[n+1]}), \ldots, f(x_n + c_s h, Y_s^{[n+1]})]^T,$$

$$c = (c_1, c_2, \ldots, c_s)^T,$$

and I is the identity matrix.
We recall also that the general linear method (1.3)-(1.4) is **preconsistent** if there exists a vector u such that

(1.7) $$Uu = e, \quad Vu = u,$$

where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^r$,

and we say that the method is **consistent** if is preconsistent and there exists tree vector $v \in \mathbb{R}^r$ such that

(1.8) $$Bu + Vv = u + v.$$

For applying the general linear method (1.3)-(1.4) we need the starting values $y_i^{[n]}$, $i = 1, 2, \ldots, r$, available for use in computing $y_i^{[n+1]}$, $i = 1, 2, \ldots, r$, in step number $n+1$. In the next, we suppose that there exist vectors $\alpha_k = (\alpha_{1k}, \alpha_{2k}, \ldots, \alpha_{rk})^T \in \mathbb{R}^r$, $k = 0, 1, 2, \ldots, p$ such that under the localizing assumption

(1.9) $$y^{[n]} = \sum_{k=0}^{p} \alpha_k y^{(k)}(x_n) h^k + O(h^{p+1}),$$

one has

(1.10) $$Y_i^{[n+1]} = y(x_n + c_i h) + O(h^{q+1}), \; i = 1, 2, \ldots, s,$$

(1.11) $$y^{[n+1]} = \sum_{k=0}^{p} \alpha_k y^{(k)}(x_n + h) + O(h^{p+1}).$$

In this situation we say that general linear method (1.3)- (1.4) has the **stage order** q and the **order** p.

The aim of this paper is to derive a family of DIMSIM methods of order $p=3$ and stage order $q=3$ by a suitable choice of the matrices A, B, U, V.

2. A family of DIMSIM methods of type 2

We recall the Theorem 3.1, [3]: If the method (1.3)-(1.4) satisfy

(2.1) $$e^{cz} = zAe^{cz} + Uw + O(z^{p+1}),$$

(2.2) $$e^{z}w = zBe^{cz} + Vw + O(z^{p+1}),$$

where

$$e^{cz} = (e^{c_1 z}, e^{c_2 z}, \dots, e^{c_s z})^T \quad \text{and}$$

(2.3) $$w = \sum_{k=0}^{p} \alpha_k z^k, \quad w = (w_1, w_2, \dots, w_r)^T,$$

then the method has the order p and the stage order $q=p$.
Using this theorem with $s=r=2$, $p=q=3$ and choosing matrices A, B, U, V of the form

(2.4) $$A = \begin{pmatrix} \lambda & 0 \\ a & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad U = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 & v_2 \\ v_1 & v_2 \end{pmatrix},$$

the order conditions (2.1)-(2.2) are

(2.5) $$e^{cz} = zAe^{cz} + Iw + O(z^4),$$

(2.6) $$e^{z} = zBe^{cz} + Vw + O(z^4),$$

where $c=(0,1)^T$, $e^{cz}=(1,e^{z})^T$, $w=(w_1,w_2)^T$, $w_i = \sum_{k=0}^{3} \alpha_{ik} z^k$, $i=1,2$,

$\alpha_k = (\alpha_{1k}, \alpha_{2k})^T$, $k=0,1,2,3$.

LEMMA 2.1. If the general linear method (1.3)-(1.4) with matrices A, B, U, V of the form (2.4) has $p=q=3$, then the order conditions (2.5),(2.6) are equivalent to the algebraic system

(2.7) $$a + (b_{21} - b_{11}) + (b_{22} - b_{12}) = 1,$$

(2.8) $$a + \lambda + (b_{22} - b_{12}) = \frac{3}{2},$$

(2.9) $$a + 3\lambda + (b_{22} - b_{12}) = \frac{7}{3},$$

(2.10) $\qquad 12v_1 + 7v_2 = 12,$

(2.11) $\qquad 12(b_{11} + b_{12}) - 5v_1 + (7 - 12a)v_2 = 7,$

(2.12) $\qquad 24b_{12} + 7v_2 = 2,$

(2.13) $\qquad 36b_{12} + 7v_2 = -3.$

Proof. From (2.5),(2.6) by developing in power series for $w = (w_1, w_2)^T$ one has,

(2.14) $\qquad w_1 = 1 - \lambda z + O(z^4),$

(2.15) $\qquad w_1 = e^z - az - \lambda z e^z + O(z^4),$

(2.16) $\qquad e^z w_1 = b_{11} z + b_{12} z e^z + v_1 w_1 + (1 - v_1) w_2 + O(z^4),$

(2.17) $\qquad e^z w_2 = b_{21} z + b_{22} z e^z + v_2 w_1 + (1 - v_2) w_2 + O(z^4)$

The relations (2.16),(2.17) by substraction, give

$$e^{2z} - e^z = ze^z(a - \lambda) + \lambda z e^{2z} + (b_{21} - b_{11})z + (b_{22} - b_{12})ze^z + O(z^4)$$

Using the power expansion of e^z and equating the coefficients in the both sides, we obtain the equations (2.7),(2.8),(2.9).

If we replace w_i from (2.14) in (2.16) and equating again the coefficients of the both sides, then we find the equations (2.10) - (2.13).

THEOTEM 2.1. There exists a family of DIMSIM methods of type 2 depending on one parameter, having $s=r=2$ and $p=q=3$, characterized by the matrices

$$A = \begin{pmatrix} \frac{5}{12} & 0 \\ a & \frac{5}{12} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{12}{7}a & -\frac{5}{12} \\ \frac{12}{7}a - \frac{1}{12} & \frac{2}{3} - a \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \frac{12}{7} \\ 0 & \frac{12}{7} \end{pmatrix}.$$

Proof. The conclusion follows withont difficulty by solving the algebraic linear system (2.7)-(2.13).

REMARK 2.1. For $a=0$ we find a type four DIMSIM method of order $p=3$, given by the matrices

$$A=\begin{pmatrix}\frac{5}{12} & 0\\ 0 & \frac{5}{12}\end{pmatrix},\ B=\begin{pmatrix}0 & -\frac{5}{12}\\ -\frac{1}{12} & \frac{2}{3}\end{pmatrix},\ U=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ V=\begin{pmatrix}0 & \frac{12}{7}\\ 0 & \frac{12}{7}\end{pmatrix}.$$

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