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LOCAL NUMERICAL STABILITY OF THE COLLOCATION METHODS FOR VOLTERRA INTEGRAL EQUATIONS

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1 Introduction

In the papers [2] and [3] we have presented a method for the construction of an approximation to the solution of the following nonlinear Volterra integral equation of the second kind:

$$(1.1)\qquad y(t) = f(t) + \int_0^t K(t,s,y(s))\,ds, \quad t \in I := [0,T],$$

with the given functions $f : I \to R$ and $K : S \times R \to R$ with $S := \{(t,s) : 0 \leq s \leq t \leq T\}$, which are supposed to be sufficiently smooth so that the integral equation (1.1) would have a unique solution $y \in C^{\alpha}(I)$, with $\alpha \in N$.

In order to describe this method let $\Pi_N : 0 = t_0 < t_1 < \ldots < t_N = T$ $\left(\text{with } t_n = t_n^{(N)}\right)$ be a quasi uniform mesh for the given interval I, and set

$$\sigma_0 := [t_0, t_1], \; \sigma_n := (t_n, t_{n+1}], \; n = 1, 2, \ldots, N-1,$$

$$h_n := t_{n+1} - t_n, \; h = \max_{(n)} h_n, \; n = 0, 1, 2, \ldots, N-1,$$

$$Z_N := \{t_n : n = 1, \ldots, N-1\}, \; \overline{Z_N} = Z_N \cup \{T\}.$$

Moreover, let $\mathcal{P}_k$ denote the space of (real) polynomials of degree not exceeding k. We then define, for given integers m and d with $m \geq 1$ and $d \geq -1$,

$$S^{(d)}_{m+d}(Z_N) := \{u : u(t)\,|_{t \in \sigma_n} =: u_n(t) \in \mathcal{P}_{m+d}, \; n = 0, \ldots, N-1,$$

$$u^{(j)}_{n-1}(t_n) = u^{(j)}_n(t_n) \text{ for } j = 0, 1, \ldots, d \text{ and } t_n \in Z_N\},$$

to be the space of polynomial splines of degree $m+d$ whose elements possess the knots Z_N and are d times continually differentiable on I. If $d=-1$, then the elements of $S_{m-1}^{(-1)}(Z_N)$ may have jump discontinuity at the knots Z_N.

An element $u \in S_{m+d}^{(d)}(Z_N)$ has for all $n=0,...,N-1$ and for all $t \in \sigma_n$, the following form:

$$(1.2) \qquad u(t) := u_n(t) = \sum_{r=0}^{d} \frac{u_{n-1}^{(r)}(t_n)}{r!}(t-t_n)^r + \sum_{r=1}^{m} a_{n,r}(t-t_n)^{d+r}.$$

From (1.2) we see that an element $u \in S_{m+d}^{(d)}(Z_N)$ is well defined when we know the coefficients $\{a_{n,r}\}_{r=\overline{1,m}}$ for all $n=0,...,N-1$. In order to determine these coefficients we consider the set of collocation parameters $\{c_j\}_{j=\overline{1,m}}$, where $0 < c_1 < ... < c_m \leq 1$, and we define the set of collocation points by:

$$X(N) = \bigcup_{n=0}^{N-1} X_n, \ with \ X_n := \{t_{n,j} := t_n + c_j h_n, \ j = 1, 2, ..., m\}.$$

The approximate solution $u \in S_{m+d}^{(d)}(Z_N)$ will be determined imposing the condition that u satisfies the Volterra integral equation (1) on $X(N)$:

$$u(t) = f(t) + \int_0^t K(t, s, u(s))\,ds, \ \ for \ all \ t \in X(N).$$

For every choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ with $0 < c_1 < c_2 < ... < c_m \leq 1$ and for all quasi-uniform mesh sequences $\{\Pi_N\}$ with sufficiently small $h > 0$, the above algorithm determines a unique approximate solution $u \in S_{m+d}^{(d)}(Z_N)$; whose convergence and local superconvergence properties have been studied in [2] and [3].

2 Numerical Stability

In order to discuss numerical stability we study the behavior of the method as applied to the integral equation, called basic test equation :

$$(2.1) \qquad y(t) = 1 + \lambda \int_0^t y(s)\,ds, \quad t \in I := [0, T],$$

λ being a constant with negative real part (see [1],[4]).

It is convenient at this stage to introduce some more notations. We write for $n = 0, 1, \ldots, N-1$:

$$\eta_n := (\eta_{n,r})^T_{r=\overline{0,d}}, \text{ where } \eta_{n,r} := \frac{u^{(r)}_{n-1}(t_n)}{r!} h_n^r, \tag{2.2}$$

and

$$\beta_n := (\beta_{n,r})^T_{r=\overline{1,m}}, \text{ where } \beta_{n,r} := a_{n,r} h_n^{d+r}. \tag{2.3}$$

With this notation an element $u \in S^{(d)}_{m+d}(Z_N)$ has for all $n = 0, \ldots, N-1$ and for any $t = t_n + \tau h_n \in \sigma_n$, the following form:

$$u_n(t_n + \tau h_n) = \sum_{r=d}^{d} \eta_{n,r} \tau^r + \sum_{r=1}^{m} \beta_{n,r} \tau^{d+r}, \quad for \;\; \tau \in [0, 1]. \tag{2.4}$$

From (2.2) and (2.3) we observe that the set of coefficients η_n is determined by the smooth conditions and the coefficients β_n are determined by collocation equations. Moreover, for $n = 0$, the values of η_0 are determined by the values of the exact solution y and its derivatives in the initial point $t = 0$.

Now, if we apply the collocation methods to test integral equation (2.1) and we use the representation (2.4) we obtain at the following collocation equation:

$$\begin{aligned}\sum_{r=1}^{m} \left(1 - \tfrac{\lambda h_n c_j}{d+r+1}\right) c_j^{d+r} \beta_{n,r} &= \sum_{r=0}^{d} \left(\tfrac{\lambda h_n c_j}{r+1} - 1\right) c_j^r \eta_{n,r} + f(t_{n,j}) + \\ &+ \sum_{i=0}^{n-1} h_i \lambda \int_0^1 u_i(t_i + \tau h_i)\, d\tau.\end{aligned} \tag{2.5}$$

for $j = 1, \ldots, m$ $(n = 0, \ldots, N-1)$

In the case when $d \geq 0$, from collocation equation (2.5) we obtain a relation between η_n and β_n:

$$V_n \beta_n = W_n \eta_n + r_n, \qquad n = 0, 1, \ldots, N-1, \tag{2.6}$$

where the matrix V_n and W_n, and the vector r_n are given by:

$$V_n := \left(\left(1 - \tfrac{\lambda h_n c_j}{d+r+1}\right) c_j^{d+r}\right)_{j,r=\overline{1,m}};$$

$$W_n := \left(w^{(n)}_{j,d}\right)_{j=\overline{1,m},\, r=\overline{0,d}};$$

$$r_n := (r_{n,j})^T_{j=\overline{1,m}};$$

with:

$$w_{j,r}^{(n)} := \begin{cases} \lambda h_n c_j, & for\ r = 0,\ j = 1,\dots,m, \\ -\left(1 - \frac{\lambda h_n c_j}{d+r+1}\right) c_j^r, & for\ r = \overline{1,\dots,d},\ j = \overline{1,\dots,m}; \end{cases}$$

$$r_{n,j} := \begin{cases} f(t_{0,j}) - f(t_0), & if\ n = 0; \\ f(t_{n,j}) - f(t_{n-1,m}) + u_{n-1}(t_{n-2,m}) - u_{n-1}(t_n) + \lambda h \int_{c_m}^{1} u_{n-1}(t_{n-1} + \tau h_{n-1}), & \\ & if\ n > 0; \end{cases}$$

By direct differentiation of relations (2.4), for the smooth conditions we get a relation between vector η_{n+1} and vectors η_n and β_n respectively:

$$\eta_{n+1} = A_n \eta_n + B_n \beta_n, \text{ for } n = 0, 1, \dots, N-2, \tag{2.7}$$

where A_n is the $(d+1) \times (d+1)$ upper triangular matrix whose $(j,r)-$element is:

$$a_{j,r}^{(n)} := \gamma_n^j . a_{j,r}, \ with\ a_{j,r} := \begin{cases} 0 & if\ r < j \\ \binom{r}{j} & if\ r \geq j \end{cases};$$

and B_n is the $(d+1) \times m$ matrix whose $(j,r)-$element is:

$$b_{j,r}^{(n)} := \gamma_n^j . b_{j,r}, \ with\ b_{j,r} := \binom{d+r}{j},$$

where $\gamma_n := \frac{h_n}{h_{n+1}}$ and $a_{j,r}$ and $b_{j,r}$ are the $(j,r)-$elements of the matrices A and B .

From the form of matrix V_n it results that for h small enough, this matrix possesses a uniform inverse for all $h_n \in (0, h)$ $(n = 0, \dots, N-1)$ and for all collocation parameters $\{c_j\}_{j=\overline{1,m}}$ with $0 < c_1 < \dots < c_m \leq 1$. Hence elimination of β_n between (2.6) and (2.7) yields:

$$\eta_{n+1} = M_n \eta_n + B_n V_n^{-1} r_n, \ with\ M_n := A_n + B_n V_n^{-1} W_n, \tag{2.8}$$

for $n = 0, \dots, N-2$.

In the case in which the mesh sequences $\{\Pi_N\}$ are uniform (i.e. $h_n = h$, $n = \overline{0, N-1}$) the matrices from (2.6), (2.7) and (2.8) do not depend on n ($\gamma_n = 1$), and the M_n are constant matrices. Under this restriction, the stability problem was discussed in paper [4], where we given the following stability criteria:

Theorem 2.1 *The polynomial spline collocation method is stable if and only if all eigenvalues of matrix $M := A + BV^{-1}W$ are in the unit disk, and any eigenvalues with $|\mu| = 1$ belongs to a 1×1 Jordan block.*

From this criteria it results that: for $d = 0$, the method is stable for all $m \geq 1$ and for every choice of the collocation parameters; for $d = 1$ the method is stable if the collocation parameters are conveniently chosen; for $d = 2$ the method is in general unstable.

The aim of the present paper is the investigation of the problem of local numerical stability of the spline collocation method.

3 Local Stability

If the mesh sequences $\{\Pi_N\}$ are quasi-uniform, i.e. there exists a finite constant γ independent of N such that $h_{max}/h_{min} < \gamma$ for all N, then the matrices M_n depend on n. In this case one may interpret the stability conditions of the above criteria as local conditions and therefore we define the notion of local numerical stability (see [1]), as:

Definition 3.1 *The spline collocation method is called locally stable at t_n if all eigenvalues of matrix M_n are in the unit disk.*

If these conditions hold in a sequence of points $t_n, \ldots, t_{n+p}$ they imply stability in the range $[t_n, t_{n+p}]$. Thus, local stability is a necessary condition for bounded propagation of isolated perturbations.

From equation (2.8) we observe that the M_n is a $(d+1) \times (d+1)$ matrix. As in paper [5], for all matrices $X_n \in \{A_n, B_n, V_n, W_n, M_n\}$, we denote by X_n^0 the matrix X_n with $h = 0$, and by μ_n^0 and μ_n the eigenvalues of M_n^0 and M_n respectively. It then follows that: $\mu_n = \mu_n^0 + O(h)$.

Concerning the local numerical stability of the spline collocation method we have the following result.

Theorem 3.2 *(i). If $d = 0$, then the polynomial spline collocation method is locally numerically stable in every point $t_n \in Z_N$ for every choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ with $0 < c_1 < \ldots < c_m \leq 1$.*

(ii). If $d = 1$, then for each set of collocation parameters there exists a constant γ_m such that the polynomial spline collocation method is locally numerically stable in the point t_n, for all $n = 0, \ldots, N-2$.

Proof. If $m_{j,r}^0(n)$ is the (j,r)-element of the matrix M_n^0 and $m_{j,r}^0$ is the (j,r)-element of the matrix M^0, where by M^0 we denote the matrix M from Theorem 2.1 with $h = 0$ (see [4]), then from the definition of these matrices we have:

$$m_{j,r}^0(n) = \gamma_n^j \, m_{j,r}^0, \quad for\ all\ j, r = 0, 1, \ldots, d$$

Thus, $\mu_n^0 = 1$ is an eigenvalue of M_n^0, for all d. In the case $d = 0$, this is the one and only eigenvalue of M_n^0, for all $n = 0, ..., N-2$. Now, the first assertion of *Theorem 3.2* follows from this remark. For prove of assertion (ii), we observe that in the case $d = 1$, the second eigenvalue of M_n^0 is given by:

$$\mu_{2,n}^0 = \gamma_n \mu_2^0, \text{ for } n = 0, ..., N-2, \tag{3.1}$$

where μ_2^0 is the eigenvalue of M^0, and it may be calculated with formula (see [5]):

$$\mu_2^0 = \frac{S_m - 2S_{m-1} + 3S_{m-2} + ... + (-1)^{m-1} m S_1 + (-1)^m (m+1)}{S_m} \tag{3.2}$$

where:

$$S_K := \sum_{1 \le i_1 < ... < i_k \le m}^{m} c_{i_1} c_{i_2} ... c_{i_k}, \quad 1 \le k \le m. \tag{3.3}$$

From (3.1) it results that for $\gamma_n \le 1$, the stability conditions obtained in [4] and [5] imply the local stability of the method in all points t_n. Because the value of μ_2^0 depends on the choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ only, by (3.1) and (3.2) it results that for each set of collocation parameters there exists a constant γ_k such that $\left|\mu_{2,k}^0\right| \le 1$. Thus, this remark ends the proof of this *Theorem*.

In the case in which the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ are the *Radau II points* for $(0, 1]$, the approximate solution $u \in S_{m+1}^{(1)}(Z_N)$ obtained by the polynomial spline collocation method, has the best local superconvergence order $2m - 1$ (see [2] and [3]). This approximation is in general numerically unstable (see [4]). The following result gives the condition in which we can have the solution u locally numerically stable.

Corollary 3.3 *If the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ are the Radau II points for $(0, 1]$, then the polynomial spline collocation method with $d = 1$ is local numerical stable in the point t_n if and only if $\gamma_n \le \frac{1}{m}$ for all $m > 1$.*

Proof. The *Radau II points* for $(0, 1]$ are the zeros of the polynomial

$$R_m(t) := P_{m-1}(2t-1) - P_m(2t-1), \tag{3.4}$$

where by P_m we denote the Legendre's polynomial of degree m. By (3.2), (3.3) and (3.4) it results that (see [5]):

$$\left|\mu_2^0\right| = \left|\frac{\left[\frac{d}{dt}(t \cdot R_m(t))\right]_{t=1}}{R_m(0)}\right| = 2\left|\frac{P'_{m-1}(1) - P'_m(1)}{P_{m-1}(-1) - P_m(-1)}\right| \tag{3.5}$$

Using the properties of Legendre's polynomials and relation (3.5) we obtain $|\mu_2^0| = w$, and thus *Corollary 3.3* follows from (3.1).

Another results regarding local numerical stability in the point t_n, may be obtained for $d > 1$. For instance, we consider the special case $d = 2$, $m = 1$ and $c_1 = 1$. This choice of d and m corresponds to the classical cubic spline functions, i.e. $u \in S_3^{(2)}(Z_N)$. In this case for the eigenvalue of M_n^0 we obtain:

$$(3.6) \qquad \mu_{1,n}^0 = 1 \quad \mu_{2,n}^0 = -\gamma_n - \gamma_n^2 + \gamma_n\sqrt{\gamma_n^2 + \gamma_n + 1}$$
$$\mu_{3,n}^0 = -\gamma_n - \gamma_n^2 - \gamma_n\sqrt{\gamma_n^2 + \gamma_n + 1}$$

From above expressions we observe that for γ_n small enough, we have $\left|\mu_{i,n}^0\right| \leq 1$ for $i = 1, 2, 3$. Thus we have:

Proposition 3.4 *The polynomial spline collocation method with $d = 2, m = 1$ and $c_1 = 1$ is locally numerically stable in the point t_n if and only if $\gamma_n \leq \frac{3}{2} - \frac{\sqrt{5}}{2} \simeq 0.381966011$.*

In the end of this section we analyze the local numerical stability of the spline collocation method in the space $S_{m-1}^{(-1)}$, for $m \geq 1$. In this case, when $d = -1$, formula (1.2) becomes:

$$(3.7) \quad u_n(t_n + \tau h_n) = \sum_{r=1}^{m} \beta_{n,r}\tau^{r-1}, \text{ for } \tau \in [0,1], \text{ and } n = 0, 1, \ldots, N-1.$$

Thus all the coefficients of the approximation $u \in S_{m-1}^{(-1)}$ are determined by the collocation conditions:

$$(3.8) \qquad \begin{aligned} V_0\beta_0 &= [f(c_1h_0), \ldots, f(c_mh_0)]^T, \quad \textit{for } n = 0, \textit{ and,} \\ V_n\beta_n &= (1, \ldots, 1)u_{n-1}(t_n) + r_n \qquad \textit{for } n = 1, 2, \ldots, N-1. \end{aligned}$$

where the matrix V_n and the vector r_n are like in relation (2.6).

If we denote by u_{n+1} the vector with m-elements $u_{n+1} := (u_n(t_n + c_jh_n))_{j=1,m}^T$, then from equation (3.7) we obtain:

$$(3.9) \qquad u_{n+1} = E \cdot \beta_n, \text{ for } n = 0, \ldots, N-1,$$

with the matrix E defined by $E := \left(c_j^{r-1}\right)_{j,r=\overline{1,m}}$. Because $V = E + O(h_n)$, the elimination of β_n between (3.8) and (3.9) yields:

$$(3.10) \qquad u_{n+1} = [1, \ldots, 1]^T (I + O(h_n))u_{n-1}(t_n) + (I + O(h_n))r_n.$$

Concerning local numerical stability of the spline collocation method in this case, we have:

Theorem 3.5 *If $d = -1$, then the polynomial spline collocation method is locally numerically stable in every point $t_n \in Z_N$ for every choice of the collocation parameters $\{c_j\}_{j=\overline{1,m}}$ with $0 < c_1 < \ldots < c_m \leq 1$.*

Proof. In this case by local numerical stability we understand the bounded propagation of isolated perturbations. Thus, if we have a perturbation of approximation solution $u \in S_{m-1}^{(-1)}(Z_N)$ in the point t_n, then from (3.10) it easy results that this perturbation is bounded propagation in the next point of mesh Π_N. If $c_m = 1$ then from (3.10) we obtain:

$$u_n(t_{n+1}) = (1 + O(h_n)) u_{n-1}(t_n) + (1 + O(h_n)) r_{n,m} \tag{3.11}$$

If $c_m < 1$, then an approximation $u \in S_{m-1}^{(-1)}(Z_N)$ can be written:

$$u_n(t_n + \tau h_n) = \sum_{j=1}^{m} L_j(\tau) \cdot u_n(t_{n,j}) \; , \quad \text{for } n = 0, 1, \ldots, N-1, \tau \in (0,1] \tag{3.12}$$

where:

$$L_j(\tau) = \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{(\tau - c_i)}{(c_j - c_i)} \, ,$$

are the fundamental Lagrange's polynomials of degree $m - 1$ and $\{c_j\}_{j=\overline{1,m}}$ are the collocation parameters. From relations (3.10) and (3.12) it results:

$$u_n(t_{n+1}) = (1 + O(h_n)) \left(u_{n-1}(t_n) + \sum_{j=1}^{m} L_j(1) r_{n,j} \right). \tag{3.13}$$

Thus the assertion of *Theorem 3.5* follows from (3.11) and (3.13) respectively, using the expression of the vector r_n.

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