Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol.XIII(1997), 67-76

A DUAL ANALYSIS OF A PROBLEM MODELLING A PLASTIC MATERIAL WITH HARDENING

CONSTANTIN GHITA

INTRODUCTION

We analyse the quasi-static equilibrium of a plastic body governed by a nonlinear subdifferential constitutive law; the directional linear hardening is expressed in terms of an internal variable: the dimensionless plastic work.

A body, occupying a regular domain Ω of R^3, is clamped on a part Γ_u of its boundary and submitted to given forces F on another part Γ_F and to unilateral contact conditions with dry friction against a rigid obstacle on the part Γ_S of the boundary. At initial state the body has a null configuration, that is $u(0)=u'(0)=$ $=0$. During a cvasi-static deformation process, the body is subjected to a uniform loading: body forces b in Ω, surface impulsive traction $F=\gamma_f \vec{t}, \left\|\vec{t}\right\|=1$, on the part Γ_F of the boundary.

Then we seek the loading factor γ_f on Γ_F and a solution (u,σ,θ^*) of the abstract governing equations modelling the local bihaviour of a plastic material with hardening:

. geometric equation:
$\dot{\varepsilon}=A_t(u)\dot{u}$ in Ω, $u=0$, $\dot{u}=0$ on Γ_u, $\forall t\in[0,T]$;
. equilibrium equation:
$A_t^*(u)\sigma=b$ in Ω, $A_t^*(u)\sigma=\gamma_f\vec{t}$ on Γ_F, $\forall t\in[0,T]$;
. unilateral contact with Coulomb′s friction:
$-R\in\partial\Phi(\dot{u})$ on Γ_S, $\forall t\in[0,T]$, $R=A_t^*(u)\sigma$ on Γ_S;
. constitutive equation (with linear hardening):
$\left(\dot{\varepsilon},-\dot{\theta}\right)\in\partial\Psi^c(\sigma,\theta^*)$, $\theta^*=H\theta$ in Ω, $\forall t\in[0,T]$.

We formulate a variational inequality on a time-independent statical admissible set of a variational space.

1. CLASSICAL FORMULATION

Let B be the current configuration of a plastic body referred to a rectangular cartesian system (Euler system): $\{x_i / i = 1, 2, 3\}$, Ω an open, bounded, connected subset of R^3, with a boundary Γ (a Lipschitz boundary is sufficient). It is decomposed into three mutually disjoint parts Γ_u, Γ_F and Γ_S; we assume that on Γ_u the *displacements* are given, on Γ_F the *surface tractions* are given and that on Γ_S *subdifferential conditions* are given.

Let U be the space of admissible displacements and $\dot{U}$ the space of admissible velocities. The space $\dot{U}$ is paired with the admissible forces space F by a bilinear form, expressed by $(\dot{u}, f)_e$, the density of external power. Let E be a strain Green space, Σ the dual space of E, the Kirchhoff stress space. A strain tensor ε is paired with a stress tensor σ by the bilinear form $(\varepsilon, \sigma) = \varepsilon : \sigma$. For a large deformation, we take $\dot{E}$ the admissible strain rate space, the dual pair of $\dot{\varepsilon} \in \dot{E}$ and $\sigma \in \Sigma$ is given $(\dot{\varepsilon}, \sigma)_i = \dot{\varepsilon} : \sigma = Tr(\dot{\varepsilon}\sigma) = \dot{\varepsilon}_{ij}\sigma_{ij}$, the density of internal power.

2. NONLINEAR GEOMETRIC MAPPING

Let $\varepsilon(v) \in E$ be the Green-Saint-Venant tensor, $\varepsilon(v) = \frac{1}{2}(\nabla v + \nabla v^T + \nabla v \nabla v^T)$, for all $v \in U$. The directional derivative of ε at $u \in U$ in the direction $v \in U$ is

$$\frac{d\varepsilon}{dv}(u) = \lim_{t \to 0_+} \frac{\varepsilon(u+tv) - \varepsilon(u)}{t} = \frac{1}{2} \lim_{t \to 0_+} \Big[\frac{\nabla(u+tv) - \nabla u}{t} + \frac{\nabla(u+tv)^T - \nabla u^T}{t} + \frac{\nabla(u+tv)\nabla(u+tv)^T - \nabla u \nabla u^T}{t}\Big] = \frac{1}{2}[\nabla v + \nabla v^T + \nabla v \nabla u^T + \nabla u \nabla v^T],$$

which emphasize the nonlinear operator - *tangent geometric mapping*

$$A_t(u) : \dot{U} \to \dot{E}, \; A_t(u)v = \frac{1}{2}[\nabla v + \nabla v^T + \nabla u \nabla v^T + \nabla v \nabla u^T],$$

then the material derivative of ε is

$$A_t(u)\dot{u} = \frac{d}{dt}\varepsilon(u) = \dot{\varepsilon}(u) = \frac{1}{2}\left[\nabla\dot{u} + \nabla\dot{u}^T + (\nabla u)(\nabla\dot{u})^T + (\nabla\dot{u})(\nabla u)^T\right]$$
$$= \left[\left(I + (\nabla u)^T\right)\nabla\right]_{sym}\dot{u}.$$

By this we mean that A_t is an affine map; we observe that it does not depend on u, in other words, $\delta^2 A_t = 0$, that is the second derivative of the orginal $\varepsilon(u)$ is a constant (symmetric) linear map.

We define its conjugate operator $A_t^*(u)$ by a Gaussian transformation, $\langle A_t(u)\dot{u}, \sigma\rangle_\Omega = (\dot{u}, A_t^*(u)\sigma)_{\bar{\Omega}}$,
where $\langle .,.\rangle_\Omega = \int_\Omega (.,.)dx$, $(.,.)_{\bar{\Omega}} = \int_\Omega (.,.)dx + \int_\Gamma (.,.)ds$.
But $\int_\Omega \left\{\frac{1}{2}\left[\nabla\dot{u} + \nabla\dot{u}^T + (\nabla u)(\nabla\dot{u})^T + (\nabla\dot{u})(\nabla u)^T\right]\right\} : \sigma dx =$(taking into account the symmetry of stress tensor) $= -\int_\Omega \dot{u}\left[\left(I + (\nabla u)^T\right)\nabla\sigma\right]dx +$
$+\int_\Gamma \left(I + (\nabla u)^T\right)\sigma\vec{n}\, ds$, where $\vec{n}$ is the outward normal vector on Γ, then $A_t^*(u) : \Sigma \to F$ is defined by $A_t^*(u)\sigma = -\left(I + (\nabla u)^T\right)\nabla\sigma$ in Ω; $=$
$= \left(I + (\nabla u)^T\right)\sigma\vec{n}$, on Γ in the sens of trace space.

3. SUBDIFFERENTIAL CONDITIONS FOR UNILATERAL CONTACT WITH DRY FRICTION

The displacements on a part Γ_S of the boundary deal with an inequality formulation. Let K be a subset of U, admissible displacements of every boundary point on Γ_S. We assume that there exists a scalar function $g : U \to \mathbb{R}$, such that any displacement u belongs to K, if and only if $g(u) \geq 0$, for all time of loading. For a displacement $u \in K$ of a boundary point on Γ_S we associate $V_K(u)$ the set of admissible velocities of a frontier point,

$$V_K(u) = \left\{v \in \dot{U} / \nabla g(u)v \in \mathbb{R}g(u) + \mathbb{R}_+\right\}.$$

We remark that $\dot{u} \in V_K(u)$, if and only if there exists $\alpha \in \mathbb{R}, \beta \in \mathbb{R}_+$, such that $\frac{d}{dt}g(u)(t) = \alpha g(u(t)) + \beta$, that is $g(u)(t) = (\alpha g(u)(0) + \beta)e^{\alpha t} - \frac{\beta}{\alpha}$.

An index T or N for the velocity $\dot{u}$ or the reaction R designate the tangential or normal component.

Suppose, for instance, that generalized *Signorini-Fichera* boundary condition hold on Γ_S, expressing the contact with a unilateral support, having a nonlinear intensity g:

$$g(u) \geq 0, R_N(u) \geq 0, \ g(u)R_N(u) = 0,$$

where $R = (I + \nabla u)\sigma n$ is the traction vector on the boundary Γ, a reaction of the obstacle.

If Coulomb′s friction law hold in Γ_S with given normal forces R_N, then $|R_T| \leq \mu|R_N|$, more precissely, if $|R_T| < \mu|R_N|$, then $\dot{u}_T = 0$ and if $|R_T| = \mu|R_N|$, then there exist $\lambda \geq 0$, such that $\dot{u}_T = -\lambda R_T$, where μ is the *friction coefficient*. We take

$$\Phi_N^*(\tau_N) = 0, \ if \ \tau_N \geq 0; = +\infty, \ otherwise$$

and seek a displacement u for which

$$\Phi_N(g(u)) = \sup_{\tau_N \in F} \{(g(u), \tau_N) - \Phi_N^*(\tau_N)\} = \sup_{\tau_N \in F_0} (g(u), \tau_N) \ ,$$

where F_0 is a convex subset of F given by $F_0 = \{\tau_N / \ \tau_N \geq 0 \text{ on } \Gamma_S\}$, the subset of unknown admissible boundary tractions τ_N.

The contact conditions (2.1) are summarized in inclusion form:

$$R_N \in \partial\Phi_N(g(u)) = \psi_K(u) = 0, \ if \ g(u) \geq 0; = +\infty, \ otherwise.$$

Also, we take

$$\Phi_T(\dot{u}_T) = \int_\Gamma \mu|R_N||\dot{u}_T| ds$$

and its conjugate functional is

$$\Phi_T^*(\tau_T) = 0, \ if \ |\tau_T| \leq \mu|R_T|; = +\infty, \ otherwise = \psi_{F_1}(\tau_T),$$

where ψ_D is the indicator function of the subset D. Here

$$F_1 = \{\tau_T / |\tau_T| \leq \mu|R_T| \ on \ \Gamma_S\} \ ,$$

then the Coulomb′s friction law is equivalent with

$$-R_T \in \partial\Phi_T(\dot{u}_T).$$

For our analysis we put $\psi_{V_K(u)}$ the indicator function of the subset $V_K(u)$ of admissible velocities. We have

Lemma 1: Let $u \in W^{1,1}([0, T]; \mathrm{R}^3)$, then the contact conditions (2.1) are equivalent to the subdifferential expression

$$-R_N(u) \in \partial \psi_{V_K(u)}(\dot{u}).$$

In the sequel we emphasize that Coulomb′s friction law take a variational formulation

Proposition 1: Let $u \in W^{1,1}([0, T]; \mathrm{R}^3)$ a displacement of a boundary point; then Coulomb′s friction law are equivalent to a variational inequality

$$R_T(u)(v_T - \dot{u}_T) + \mu |R_N| (|v_T| - |\dot{u}_T|) \geq 0, \forall v \in V_K(u).$$

The last relation is an explicit writing of the subdifferential inclusion

$$-R_T(u(t)) \in \partial \Phi_T(\dot{u}(t)).$$

Summarizing the last two results we deduce the following

Proposition 2: The behaviour of the body on the part Γ_S of the boundary Γ taking into account both the unilateral contact and the Coulomb′s friction law, is caracterized by the inequality

$$\begin{aligned} & u(o) \in K, \\ & R(u(t))(v - \dot{u}(t)) + \mu |R_N(u(t))| (|v_T| - |\dot{u}(t)_T|) \geq 0, \\ & \forall v \in V_K(u(t)), \text{a.e. } t \in [0, T]. \end{aligned}$$

Remark: If we take the functional $\Phi(v) = \int_\Gamma \mu |R_N| |v_T| ds$ as an external power developed on the Γ_S, we give subdifferential conditions on Γ_S,

$$-R(t) \in \partial \Phi(\dot{u}(t)), \forall t \in [0, T].$$

This last relation is equivalent to the following

$$\dot{u}(t) \in \partial \Phi^c(-R(t)), \forall t \in [0, T].$$

4. CONSTITUTIVE NONLINEARITY OF A PLASTIC BODY WITH STRAIN HARDENING

Denote by θ an internal variable, say a time function, for example we may take $\theta(t) = \frac{1}{m(\Omega)\sigma_c}\int_0^t (\sigma, A_t^*(u)\dot{u})_\Omega\, dt$, $\theta(0) = 0$, as a dimensionless plastic power, where σ_c is a material constant, say the yield threshold at a deformation by traction. Let Θ be the space of admissible hardening factor and Θ^* the space of θ^*, conjugate function of $\theta \in \Theta$. For a material with linear hardening we assume $\theta^* = H\theta$, where H is a positive factor.

Let $K \subset \Sigma \times \Theta^*$ be a convex nonempty subset

$$K = \{(\sigma, \theta^*) \in \Sigma \times \Theta^* / \varphi(\sigma, \theta^*) \leq 0 \text{ in } \Omega\},$$

where $\varphi(\sigma, \theta^*) = T(\sigma) - \eta(\theta^*) - \sigma_c$, T is a convex lower semicontinuous function of the stress tensor - the plastic yielding function, η is a hardening monotone function. We introduce the indicator function of convex set K,

$$\psi_K(\sigma, \theta^*) = 0, (\sigma, \theta^*) \in K; = +\infty, \textit{ otherwise},$$

it is convex, lower semicontinuous and subdifferentiable, and $\partial\psi_K(\sigma, \theta^*)$ is a convex subset of $\dot{E} \times \Theta$, ψ_K is so-called complementary plastic superpotential. The duality between $\dot{E} \times \Theta$ and $\Sigma \times \Theta^*$ is given by (see also [6]):

$$\left\langle (\sigma, \theta^*), (\dot{\varepsilon}, -\dot{\theta}) \right\rangle = (\sigma, A_t(u)\dot{u}) + \left\langle \theta^*, -\dot{\theta} \right\rangle_T,$$

where

$$\left\langle \theta^*, -\dot{\theta} \right\rangle_T = \int_0^T \theta^*(t)\left(-\dot{\theta}(t)\right) dt = -\theta(T)\theta^*(T) + \left\langle \dot{\theta}^*, \theta \right\rangle_T.$$

Consider ψ_K^* the support function of the convex subset K, $\psi_K^* : \dot{E} \times \dot{\Theta} \to \mathrm{R}$,

$$\psi_K^*\left(\dot{\varepsilon}, -\dot{\theta}\right) = \sup_{(\sigma,\theta^*)\in\Sigma\times\Theta^*} \left\{ \left\langle (\sigma, \theta^*), \left(\dot{\varepsilon}, -\dot{\theta}\right) \right\rangle - \psi_K(\sigma, \theta^*) \right\}$$
$$= \sup_{(\sigma,\theta^*)\in K} \left\{ (\sigma, A_t(u)\dot{u}) + \left\langle \dot{\theta}^*, \theta \right\rangle - \theta(T)\theta^*(T) \right\}$$

The definition of the complementary plastic superpotential

$$\Psi^c(\sigma, \theta^*) = \psi_K(\sigma, \theta^*)$$

suggests in a greater generality the constitutive equation of the hardening plastic material written in a subdifferential inclusion form

$$(3.1)\ \left(\dot{\varepsilon}, -\dot{\theta}\right) \in \partial\Psi^c(\sigma,\theta^*) = \left(\dot{\lambda}\frac{\partial\varphi}{\partial\sigma}, \dot{\lambda}\frac{\partial\varphi}{\partial\theta^*}\right), \ if\ \varphi(\sigma,\theta^*) < 0 \text{ and }$$
$$\dot{\lambda} \geq 0; = (0,0),\ if\ \varphi(\sigma,\theta^*) = 0; = +\infty,\ if\ \varphi(\sigma,\theta^*) > 0$$

Then, for a given displacement $u \in U$, the constitutive relation may be written as $A_i(u)\dot{u} = \dot{\lambda}\, T'(\sigma); -\dot{\theta} = \dot{\lambda}\eta'(\theta^*)$, iff the constraints $\varphi(\sigma,\theta^*) = 0,\ \dot{\lambda} \geq 0$ are satisfied.

We use some considerations of convex analysis: the subdifferential inclusion (3.1) is equivalent to the following inequality:

$$(3.2)\ (A_i(u)\dot{u}, \tau - \sigma) + \left\langle -\dot{\theta}, \phi^* - \theta^* \right\rangle \leq \Psi^c(\tau,\phi^*) - \Psi^c(\sigma,\theta^*),$$
$$\forall(\tau,\phi^*) \in \Sigma \times \Theta^*,$$

under the same constraints: (σ,θ^*) must be on the yield surface $\varphi(\sigma,\theta^*) = 0$; Lagrange multiplier $\dot{\lambda} \geq 0$, there exist

$$(A_i(u)\dot{u}, \tau - \sigma) + \left\langle -\dot{\theta}, \phi^* - \theta^* \right\rangle \leq \Psi^c(\tau,\phi^*),$$

a generalized Drucker's postulate, and if $(\tau,\phi^*) \in K$, we have the problem:

$$\text{find } (\sigma,\theta^*) \in K, \text{ such that}$$
$$(A_i(u)\dot{u}, \tau - \sigma) + \langle \dot{\theta}, \phi^* - \theta^* \rangle \leq (\phi^*(T) - \theta^*(T))\theta(T),$$
$$\text{for all } (\tau,\phi^*) \in K.$$

5. THE GAP FUNCTION AND THE MAIN RESULT

We reconsider the inequality (3.2) and take

$$S_v = \{(v,\tau,\phi^*) \in U \times \Sigma \times \Theta^* / A_i^*(v)\tau = b, \text{ in } \Omega;$$

$$A_i^*(v)\tau - v(\tau,v)\overrightarrow{t} = 0 \text{ on } \Gamma_i\}$$

the time independent statically admissible set ; here the loading intensity is associated with the field (τ, v). We take $u = v + \delta u$, $\phi^* + \delta\theta^* = \theta^*$, assuming that v, ϕ^* are time independent.

We finish this section with a simple calculus of the Gateaux derivative of the strain rate tensor $\varepsilon(v + \delta u)$, that is

$$\frac{d}{dt}\varepsilon(u) = \dot{\varepsilon}(u) = \dot{\varepsilon}(v + \delta u) = A_l(v + \delta u)\delta\dot{u}\ (\dot{u} = \delta\dot{u}) =$$
$$= \tfrac{1}{2}\left[\nabla\delta\dot{u} + \nabla\delta\dot{u}^T + \nabla\delta\dot{u}\nabla(v + \delta u)^T + \nabla\delta\dot{u}^T\nabla(v + \delta u)\right] =$$
$$= \tfrac{1}{2}[\nabla\delta\dot{u} + \nabla\delta\dot{u}^T + \nabla\delta\dot{u}\nabla v^T + \nabla\delta\dot{u}^T\nabla v] + \tfrac{1}{2}[\nabla\delta\dot{u}\nabla\delta u^T +$$
$$+ \nabla\delta\dot{u}^T\nabla\delta u] = A_l(v)\delta\dot{u} + A_n(\delta u)\delta\dot{u},$$

where $A_n(w)k = \frac{1}{2}[\nabla k\nabla w^T + \nabla k^T\nabla w]$ is a *compensatory operator.*

6. VARIATIONAL FORMULATION

We refer to the compensatory operator; we take

$$\tfrac{d}{dt}(\nabla\delta u\nabla\delta u^T) = \nabla\delta\dot{u}\nabla\delta u^T + \nabla\delta u\nabla\delta\dot{u}^T = 2A_n(\delta u)\delta\dot{u}, \text{ and}$$
$$2\int_0^{t_f}(A_n(\delta u)\delta\dot{u}, S)_\Omega dt = \int_0^{t_f}\tfrac{d}{dt}(\nabla\delta u\nabla\delta u^T, S)_\Omega dt =$$

$= \int_\Omega \nabla\delta u\nabla\delta u^T : S dx$ (S is a vanish tensor at initial state) $= G(\delta u, S)$ $(= G(u - v, S))$, that is so-called *gap function* associated to the compensatory operator.

Coming back to the constitutive inequality (3.2) we can formulate

Theorem 1: Suppose that H^{-1} is a positive definite matrix, G a non-negative gap function, $\left(u, \vec{t}\right) > 0$ on Γ_t, then

$$v_c \geq v(v, \tau) - t_f\left[\int_\Omega \Psi^c(\tau, \phi^*) + \Psi_{fr}(-R_\tau(t))\right], \forall(v, \tau, \phi^*) \in S_s,$$
$$\forall t \in [0, t_f],$$

where Ψ^c is the complementary plastic superpotential with hardening, $\Psi^c_{fr}(-\tau) = 0$, iff $|\tau_T| \leq \mu|\tau_N|, \tau_N \geq 0$ the complementary superpotential corresponding to the contact with dry friction, $v(v, \tau)$ is an admissible intensity factor on the part Γ_t, t_f is a response time of quasi-static loading on the boundary Γ.

Proof: Taking into account the inequality (3.2) we have

$$\Psi^c(\tau,\phi^*)-\Psi^c(\sigma,\theta^*)\geq (A_l(v+\delta u)(\dot v+\delta\dot u),\tau-\sigma)+\left(-\dot\theta,\phi^*-\theta^*\right)$$

$=(A_l(v)\delta\dot u,\tau)+(A_n(\delta u)\delta\dot u,\tau)-(A_l(u)\delta\dot u,\sigma)$ (having in mind the linear hardening $\theta^*=H\theta,\ \theta=H^{-1}\theta^*\Rightarrow\dot\theta=-H^{-1}\delta\dot\theta^*$) $+\left(H^{-1}\delta\dot\theta^*,\delta\theta\right)$.

By an integration on Ω and using the Gauss-Green transformation, we obtain

$$\int_\Omega\Psi^c(\tau,\phi^*)dx-\int_\Omega\Psi^c(u,\sigma)dx\geq(A_l(v)\delta\dot u,\tau)_\Omega+$$
$$+(A_n(\delta u)\delta\dot u,\tau)_\Omega-(A_l(u)\delta\dot u,\sigma)+\int_\Omega\left(H^{-1}\delta\dot\theta^*,\delta\theta^*\right)dx=$$
$$=(\delta\dot u,A_l^*(v)\tau)_\Omega+(\delta\dot u,A_l^*(v)\tau)_\Gamma-(\delta\dot u,A_l^*(u)\sigma)_\Omega-(\delta\dot u,A_l^*(u)\sigma)_\Gamma+$$
$$+\int_\Omega(A_n(\delta u)\delta\dot u,\tau)dx+\int_\Omega\left(H^{-1}\delta\dot\theta^*,\delta\theta^*\right)dx=\int_\Omega A_n(\delta u)\delta\dot u:\tau dx+$$
$$+\left(\delta\dot u,(v(v,S)-v_c)\overrightarrow{t}\right)+(\delta\dot u,A_l^*(v)\tau)_{\Gamma_S}-(\delta\dot u,A_l^*(u)\sigma)_{\Gamma_S}+$$
$$+\left(H^{-1}\delta\dot\theta^*,\delta\theta^*\right)_\Omega.$$

By performing the time integration in the interval $[0,t_f]$, we have

$$\int_0^{t_f}dt\int_\Omega\Psi^c(v,\tau)dx-\int_0^{t_f}dt\int_\Omega\Psi^c(u,\sigma)dx\geq$$
$$\int_0^{t_f}dt\int_\Omega A_n(\delta u)\delta\dot u:\tau dx+\int_0^{t_f}\left(\dot u,(v(v,S)-v_c)\overrightarrow{t}\right)dt+$$
$$+\int_0^{t_f}(\dot u,A_l^*(v)\tau-A_l^*(u)\sigma)_{\Gamma_S}dt+\int_0^{t_f}(H^{-1}\delta\dot\theta^*,\delta\theta^*)_\Omega dt.$$

Now we are refering to the compensatory operator; we evaluate the term expressing the contact conditions with dry friction on the boundary Γ_S. Finally, we focus attention to differential inclusion (3.5):

$$\int_0^{t_f}(\dot u,A_l^*(v)\tau-A_l^*(u)\sigma)_{\Gamma_S}=-\int_0^{t_f}(\dot u,R_\tau(t)-R_\sigma(t))_{\Gamma_S}dt\geq$$
$$\geq-\int_0^{t_f}\Psi^c(-R_\tau(t))dt+\int_0^{t_f}\Psi^c(-R_\sigma(t))dt$$

We have in mind the analytic expression of the superpotential Ψ^c and that R_σ satisfies the contact and friction conditions and $\Psi^c(-R_\sigma)=0$, but $(u,\sigma,\theta)\in S_2$, then $\Phi^c(u,\sigma)=0$, also τ,ϕ^* are time-independent fields and τ,ϕ^*, v are vanish at the initial state. Summarizing we have

$$t_f \int_{\Omega} \Psi^c(\tau, \phi^*)dx \geq G(v-u,\tau) + (v(v,\tau) - v_c)\left(u, \overrightarrow{t}\right)_{\Gamma_t} + \\ +\tfrac{1}{2}(\phi^* - \theta^*, H^{-1}(\phi^* - \theta^*)).$$

It is a simple rearragement which gives the inequality of the theorem.

The last result gives an inferior bound for the intensity impulsive factor on the boundary Γ_t, which a tractio is acting.

REFERENCES

[1] Moreau J. J., *On unilateral constraints, friction and plasticity*, in: "New Variational Techniques in Mathematical Physics, G. Capriz & G. Stampacchia, Ed. Cremonese, Roma, 1973

[2] Cimetiere A., Leger A. & Naejus C., *A functional framework for the Signorini problem with Coulomb's friction*, EDF, Directions des Etudes et Recherches, 1994

[3] Gao Y & Strang G, Geometric nonlinearity potential energy, complementary energy, and the gap function. Quat. of Appl. Math. XLVII, 1989, p. 487-504

[4] Gao Y & Wierzbicki T., Bounding theorem in finite plasticity with hardening effect, Quat. Appl. Math., XLVII, 1989, p. 395-403

[5] Gao Y, Dynamically loade rigid-plastic analysis under large deformation, Quat. Appl. Math., XLVIII, 1990, p.731-739

[6] Ghiţă C, Metalurgie MatematicĂ. Analiza nestandard a proceselor metalurgice, Ed. Academiei Romane, 1995

[7] Panagiotopoulos P. D., *Multivalued Boundary Integral Equations for Inequality Problems. The convex case*, Acta Mechanica 70, 1987, p.145-167

Received 17.10.1996

University " Valachia" of Târgovişte
Calea Domnească 236
0200 Târgovişte
ROMANIA