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THE INVARIANCE TO HOMOTOPY OF THE TOPOLOGICAL DEGREE IN SPACE WITH SEMI-INNER PRODUCT

Cristinel MORTICI

Abstract. *Some results in topological degree theory are given for spaces with inner product (Hilbert spaces). We give a method to extend these results in generally normed spaces using the semi-scalar product introduced by G. Lumer. Results on existence of eigenvalues, on fixed point theory and on surjectivity for some nonlinear operators are given.*

To each pair $\{x, y\}$ of a real normed linear space X, we can associate a real number $[x, y]$ such that

$[x+y, z] = [x, z]+[y, z]$, $[\lambda x, y] = \lambda[x, y]$, $[x, x] = \|x\|^2$, $|[x, y]| \leq \|x\|\cdot\|y\|$.

$[x, y]$ is called a semi-scalar product of the vectors x and y. Indeed, this semi-scalar product is given by $[x, y] := \varphi_y(x)$, with φ_y is arbitrary chosen from $J(y)$, for each $y \in X$, where $J : X \to X^*$ is the duality map.

From now X will be a real normed space endowed with a semi-scalar product $[x, y]$, $\dim X < \infty$. Suppose that $D \subset X$ is a bounded domain containing the origin. Similar we can apply the next results also for an infinit dimensional normed space, but the operators which we work with must be compact perturbations of the identity, because we deal with Leray-Schauder degree in this case.

For the begining we give the following:

Theorem 1. *Let $f \in C(\overline{D})$ such that $[f(x),x]\geq 0$ on ∂D. Then there exists $x_0 \in \overline{D}$ such that $f(x_0) = 0$.*

Proof: Let us consider the homotopy $h_t(x) = (1-t)f(x)+tx$. If $h_t(x) = 0$ for some $x \in \partial D$ and $t \in [0, 1]$ we obtain, applying the semi-scalar product:

$$(1-t)[f(x), x] + t\,\|x\|^2 = 0,$$

therefore $x = 0 \in \partial D$, contradiction.

Then we can define $d(h_t, D, 0)$ and from the invariance to homotopy of the topological degree it follows

$$d(f, D, 0) = d(h_0, D, 0) = d(h_1, D, 0) = d(I, D, 0) = 1 \neq 0,$$

i.e. the equation $f(x) = 0$ has solution in D. ■

Observation: The condition " $[f(x), x] \geq 0$" can be replaced with " $[f(x), x]$ does not change the sign on ∂D" because it can be repeated the proof for $-f$.

Theorem 2. *Let $f \in C(\overline{D})$ such that $\inf_{\|x\|=a} [f(x), x] > 0$ for some $a > 0$. Then there exists $\lambda > 0$ such that $f(x) = \lambda x$.*

Proof: Let us suppose that $[f(x), x] \geq m > 0$, $\forall\ \|x\| = a$, which can be written as

$$[f(x), x] \geq \frac{m}{a^2} \|x\|^2 ,$$

or

$$[f(x), x] \geq \lambda \|x\|^2 , \text{ with } \lambda = \frac{m}{a^2}$$

and $[f(x) - \lambda x, x] \geq 0$, $\forall\ x \in \partial B(0, a)$. The conclusion follows from theorem 1. ■

Theorem 3. *Let $f \in C(X)$ such that $\sup_{x \in X} \|x - f(x)\| = M < \infty$. Then there exists $\lambda \neq 0$ and $x \in X$ with $f(x) = \lambda x$.*

Proof: Let $n > 0$ such that $n - \frac{1}{n^2} > M$. We shall prove that the conditions from theorem 2 are satisfied for $a = n$. If we suppose that $\inf_{\|x\|=n} [f(x), x] \leq 0$, then there exists $\|x_0\| = n$ with

$$[f(x_0), x_0] < \frac{1}{n} \Leftrightarrow [f(x_0) - x_0, x_0] + \|x_0\|^2 < \frac{1}{n} \Rightarrow$$
$$\Rightarrow n^2 - \frac{1}{n} < [x_0 - f(x_0), x_0] \leq \|x_0 - f(x_0)\| \cdot \|x_0\| \Rightarrow$$
$$\Rightarrow n^2 - \frac{1}{n} \leq n \cdot M \Rightarrow n - \frac{1}{n^2} \leq M,$$

contradiction. Now we can apply theorem 2, because $\inf_{\|x\|=n} [f(x), x] > 0$. ■

Theorem 4. *Let $D \subset X$ be a bounded, symmetric domain with $0 \in D$ and let $f \in C(\overline{D})$ such that $f(x) \neq f(-x)$, $\forall\, x \in \partial D$. Then for every $\lambda \neq 0$ such that λx does not belong to the line determined by $f(x)$ and $f(-x)$, $\forall\, x \in \partial D$, there exists $x_0 \neq 0$ with $f(x_0) = \lambda x_0$.*

Proof: Let us consider the homotopy $h_t(x) = f(x) - tf(-x) - (1-t)\lambda x$. Then $d(h_t, D, 0)$ is well defined. Indeed, $h_1(x) \neq 0$, $\forall\, x \in \partial D$ and if $h_t(x) = 0$ for some $t \in [0,1)$ and $x \in \partial D$, it results

$\lambda x = \dfrac{f(x) - tf(-x)}{1-t}$ which is a contradiction with the fact that λx does not belong to the line determined by $f(x)$ and $f(-x)$. Using the invariance to homotopy of the topological degree, we obtain

$d(f - \lambda I, D, 0) = d(f(x) - f(-x), D, 0) \neq 0$ because is odd (from Borsuk theorem). So there exists $x_0 \neq 0$ with $f(x_0) = \lambda x_0$. ■

Theorem 5. Let $f \in C(\overline{D})$ and $a > 0$ such that $M = \sup_{\|x\|=a} \|f(x)\| < \infty$.

Then for every $\lambda \in \mathbf{R}$, $|\lambda| > \dfrac{M}{a}$, there exists $x_0 \in X$ with $f(x_0) = \lambda x_0$.

Proof: Let us denote $B = B(0,a)$ and consider the homotopy $h_t(x) = tf(x) - \lambda x$. If $tf(x) = \lambda x$ for some $\|x\| = a \Rightarrow |\lambda| \cdot a \leq \|f(x)\| \Rightarrow$

$|\lambda| \leq \dfrac{M}{a}$, contradiction. We define $d(h_t, B, 0)$ and obviously

$d(f - \lambda I, B, 0) = d(-\lambda I, B, 0)$. ■

The following result is a variant of Krasnoselski-Shinbrot condition of fixed point (e.g. [6],p.77).

Theorem 6. *Let $D = B(0,r)$ and $f \in C(\overline{D})$ such that $[f(x), x] \leq \|x\|^2$ on ∂D. Then f has a fixed point.*

Proof: We shall use Leray-Schauder fixed point theorem with $x_0 = 0$. Indeed, if $f(x) = \lambda x$ for $x \in \partial D$ and $\lambda > 0$, then $[\lambda x, x] \leq \|x\|^2 \Rightarrow$

$\Rightarrow \lambda \|x\|^2 \leq \|x\|^2 \Rightarrow \lambda \leq 1$. ■

There is a well known surjectivity result in [2], p.19, for the case $X = \mathbf{R}^m$. More generally, we have the following:

Theorem 7. *Let $f \in C(X)$ be such that $\dfrac{[f(x), x]}{\|x\|} \to \infty$ as $\|x\| \to \infty$. Then f is onto X.*

Proof: Let $p \in X$ and $h_t(x) = tx + (1-t)f(x) - p$. At $\|x\| = r$ we have

$$[h_t(x), x] = t\|x\|^2 + (1-t)[f(x), x] - [p, x] \Rightarrow r(tr + (1-t)\frac{[f(x), x]}{\|x\|} - \|p\| > 0$$

for $t \in [0,1]$ and $r > \|p\|$ sufficiently large. Therefore, $d(f, B(0,r), p) = 1 \neq 0$ for such an r, i.e. the equation $f(x) = p$ has solutions. ■

REFERENCES:

[1] BERKOVITS,J., *On the degree theory for nonlinear Mappings of Monotone Type*. Annales Academiae Scientiarum Fennicae,58,1986.

[2] DEIMLING,K., *Nonlinear Functional Analysis*. Springer Verlag,1980.

[3] LUMER,G., *Semi-inner Product Spaces*. Trans.Amer.Math.Soc.,29-43,1961.

[4] MORTICI,C., *Eigenvalues Problem for Laplacian with Boundary Conditions of Neumann Type*. A 19-a Conf.de Mecanica Solidelor,vol.III,p.480-482,1995.

[5] PASCALI,D.,SBURLAN,S. *Nonlinear Mappings of Monotone Type*. Sijthoff&Noordhoff Intern.Publ.,Alphen aan den Rijn,1978.

[6] SBURLAN,S. *Gradul Topologic*. Ed.Acad.,Bucuresti,1983.

[7] Yosida,K. *Functional Analysis. Springer Verlag,1965.*

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Cristinel Mortici
"OVIDIUS" University
Department of Mathematics
Bd.Mamaia 124
8700 Constanta - ROMANIA