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APPROXIMATION PROPERTIES OF CESARO MEANS FOR FOURIER SERIES OF INTEGRABLE FUNCTIONS

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Summary. Deviation of Cesaro (C,α) means for integrable functions and these functions is represented in a form with a main term as an improper integral of second order difference for these functions and a remainder. The exact order of this remainder is obtained, that is an estimation from above and below are proved in terms of second moduli of smoothness for given function but not only the estimation from above as in the papers of other authors dealing with this problem. Detailed proofs are given.

Let a function $f \in L_p$ $(1 \le p \le \infty)$ be a 2π-periodical, and let

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx} = \sum_{k=-\infty}^{\infty} A_k(x) \qquad (1)$$

be the Fourier series for $f(x)$.

Let

$$\Delta_\delta^2 = f(x-\delta) - 2f(x) + f(x+\delta)$$

and

$$\omega_2(f,h)_p = \sup_{0<\delta\le h} \|\Delta_\delta^2 f(x)\|_p$$

be (respectively) the second order symmetric difference and the modulus of smoothness for $f(x)$ of second order.

All the norms below are in the metric of L_p.

Let us construct the following classical linear means of (1):

$$\sigma_n^\alpha(f;x) = \sum_{|k|\le n} \frac{A_{n-|k|}^\alpha}{A_n^\alpha} A_k(x),$$

$$R_n^\alpha(f;x) = \sum_{|k|\le n} \left(1 - \left(\frac{|k|}{n+1}\right)^\alpha\right) A_k(x),$$

i.e. Cesaro (C,α) and Riesz means respectively. The principal feature of these means is the fact that arithmetic means are the special case of these means if $\alpha = 1$:

$$\sigma_n(f;x) = \sigma_n^1(f;x) = R_n^1(f;x) = \sum_{|k|\le n} \left(1 - \frac{|k|}{n+1}\right) A_k(x).$$

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The differences or deviations $f(x) - \sigma_n^\alpha(f;x)$ and $f(x) - R_n^\alpha(f;x)$ were investigated by some authors in different directions. For $f(x) - \sigma_n^\alpha(f;x)$ the main term of this deviation was obtained in different situations (different α and p for example) in some papers (see [1], [2] for further references and comments as well).

Here is one of these results due to H.K. Lebed and A.A. Avdienko [1] for the mentioned functions:

$$f(x) - \sigma_n(f;x) = -\frac{1}{2\pi}\int_1^\infty \Delta^2_{\frac{t}{n+1}} f(x) t^{-2}\,dt + \delta_n(f;x), \tag{2}$$

$$\|\delta_n(f;x)\| \le C\omega_2\left(f;\frac{1}{n+1}\right)$$

with a constant C depending at most on p.

The constants C below may be of cause different at different occurrences depending only on mentioned parameters or independent of them.

The generalization of (2) for $\sigma_n^\alpha(f;x)$ is obtained by M.M. Lekishvili. in [2]. I.P. Falaleev has generalized (2) for Riesz means.

First we will state our results in this direction.

Theorem 1. *For $f \in L_p$ $(1 \le p \le \infty)$, $\alpha > 1$, there exist $C_i(p,\alpha)$ $(i = 1,2)$ such that*

$$f(x) - \sigma_n^\alpha(f;x) = \alpha(f - \sigma_n(f;x)) + \delta_n(f;x), \tag{3}$$

$$C_1\omega_2\left(f;\frac{1}{n+1}\right) \le \|\delta_n(f;x)\| \le C_2\omega\left(f;\frac{1}{n+1}\right) \tag{4}$$

Theorem 2. *For $f \in L_p$ $(1 \le p \le \infty)$, $\alpha > 1$, there exist $C_i(p,\alpha)$ $(i = 1,2)$ such that*

$$f(x) - \sigma_n^\alpha(f;x) = -\frac{\alpha}{2\pi}\int_1^\infty \Delta^2_{\frac{t}{n+1}} f(x) t^{-2}\,dt + \tau_n(f;x),$$

$$C_1\omega_2\left(f;\frac{1}{n+1}\right) \le \|\tau_n(f;x)\| \le C_2\omega_2\left(f;\frac{1}{n+1}\right). \tag{5}$$

These results were partially announced in [3]. To prove these theorems some well known auxiliary results are needed.

First we will state the comparison principle for linear means of (1). This principle was proposed by R.M. Trigub [4]. Given matrix $\Lambda = \|\lambda_k^{(n)}\|$, whose elements depend on n $(k \in Z,\ n \in N)$. We will form the linear means (operators) for (1) with the help of this matrix

$$\tau_n(f;\Lambda) = \tau_n(f;\Lambda,x) \sim \sum_{k=-\infty}^{\infty} \lambda_k^{(n)} A_k(x)$$

and suppose that the function $\tau_n(f;\Lambda,x)$ with given Fourier series belongs to L_p. The norm $\tau(\Lambda)$ of this linear operator

$$\tau(\Lambda) = \int_0^\pi \left|\sum_{k=-\infty}^{\infty} \lambda_k^{(n)} e^{ikx}\right| dx$$

is called the Lebesgue constant for the corresponding matrix.

As in a situation above with the help of another matrix $\dot{\Lambda} = \|\dot{\lambda}_k^{(n)}\|$ we will form the means

$$\dot{\tau}_n(f;\dot{\Lambda}) = \tau_n(f;\dot{\Lambda},x) \sim \sum_{k=-\infty}^{\infty} \dot{\lambda}_k^{(n)} A_k(x)$$

and let the function $\tau_n(f;\dot{\Lambda},x)$ belongs to L_p too.

Theorem A. *(Comparison principle: R.M. Trigub) The inequality holds*

$$\|\tau_n(f;\Lambda,x)\| \le r(\Lambda^*)\|\tau_n(f;\dot{\Lambda},x)\|,$$

where $r(\dot{\Lambda}^*)$ – *the norm of appropriate operator (Lebesgue constant) defined by the matrix* $\Lambda^* = \|\lambda_k^{*(n)}\|$, $\lambda_k^{*n} = \frac{\lambda_k^{(n)}}{\dot{\lambda}_n}$.

Matrix Λ^* is called the transitional matrix for the inequality above.

Theorem B. *(S.A. Telyakovskii [5]) If a sequence $\{a_k\}$ is even and satisfies the conditions:*

1) $\lim_{k\to\infty} a_k = 0$,
2) *there exists a sequence $\{A_k\}$ such that A_k tends to zero monotonically decreasing and $\sum_{k=0}^{\infty} A_k < \infty$*,
3) $|\Delta a_k| \le A_k \quad (\Delta a_k = a_k - a_{k+1})$.

then the Lebesgue constant for this sequence is bounded.

Proof of Theorem 1.

The relation (6) may be checked directly using (3)

$$\delta_n(f;x) = (1-\alpha)(f(x) - \sigma_n^2(f;x)) + \alpha\sigma_n(f;x) \qquad (6)$$

and the matrix for (6) is $\Lambda = \|\Lambda_k^{(1)(n)}\|$, where (we remind that k stands for $|k|$ in the proofs of theorems)

$$\Lambda_k^{(1)(n)} = 1 - \frac{\alpha k}{n+1} - \frac{A_{n-k}}{A_n^2} \text{ if } k \le n, \quad \Lambda_k^{(1)(n)} = 1 - \alpha \text{ if } k > n.$$

To prove (4) we prove the inequality

$$c_1\|f(x) - R_n^2(f;x)\| \le \|\delta_n(f;x)\| \le c_2\|f(x) - R_n^2(f;x)\| \qquad (7)$$

Inequality (4) follows immediately from (7), because

$$c_3\omega_2\left(f;\frac{1}{n+1}\right) \le \|f(x) - R_n^2(f;x)\| \le c_4\omega_2\left(f;\frac{1}{n+1}\right).$$

It was namely R.M. Trigub who has received the bilateral estimates like to given one in a case of linear means of Fourier series. This double estimate for $\|f(x) - R_n^2(f;x)\|$ was proved by R.M. Trigub as well [4] even in more general situation and not only for the second order moduli.

Let us start with the proof of the right-side inequality in (7). A matrix for $f(x) - R_n^2(f;x)$ is

$$\bar{\Lambda} = \|\Lambda_k^{(2)(n)}\|, \quad \Lambda_k^{(2)(n)} = \frac{k}{n+1} \ \text{ if } \ k \le n \ \text{ and } \ \Lambda_k^{(2)(n)} = 1 \ \text{ for } \ k > n$$

The transitional matrix for the right-side inequality in (7) is (see (5))

$$\Lambda^* = \|\Lambda_k^{(n)}\|, \quad \Lambda_k^{(n)} = \left(1 - \frac{\alpha_k}{n+1} - \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}}\right)\left(\frac{n+1}{k}\right)^2, \quad 0 < k \le n$$

and

$$\Lambda_k^{(n)} = 1 - \alpha \ \text{ for } \ k > n, \quad \Lambda_0^{(n)} = \alpha(1-\alpha).$$

Some elements of behavior of $\Lambda_k^{*(n)}$ are needed, but first some notations. Let $\nu_0 = \mu_0 = \lambda_0 = 0$ and for $1 \le k \le n$

$$\nu_k = 1 - \frac{\alpha_k}{n+1} - \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}},$$

$$\mu_k = \frac{\nu_k}{k} = \frac{1}{k}\left(1 - \frac{\alpha_k}{n+1} - \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}}\right),$$

$$\lambda_k = \frac{\mu_k}{k} = \frac{1}{k^2}\left(1 - \frac{\alpha_k}{n+1} - \frac{A_{n-k}^{\alpha}}{A_n^{\alpha}}\right)$$

then $\Lambda_k^{*(n)} = (n+1)^2 \lambda_k$, $1 \le k \le n$.

One may check the identities below using direct evaluations:

$$\Delta\nu_k = \frac{\alpha}{n+1} - \frac{\Delta A_{n-k}^{\alpha}}{A_n^{\alpha}}, \quad \Delta_2\nu_k = -\frac{\Delta_2 A_{n-k}^{\alpha}}{A_n^{\alpha}}, \quad \Delta_3\nu_k = -\frac{\Delta_3 A_{n-k}^{\alpha}}{A_n^{\alpha}},$$

$$\Delta_4\nu_s = -\frac{\Delta A_{n-k}^{\alpha}}{A_n^{\alpha}},$$

$$\Delta\left(\frac{\nu_k}{k}\right) = -\frac{1}{k(k+1)}\sum_{s=0}^{k-1}(s+1)\Delta_2\nu_s,$$

$$\Delta_2\left(\frac{\nu_k}{k}\right) = -\frac{1}{k(k+1)(k+2)}\sum_{s=0}^{k-1}(s+1)(s+2)\Delta_3\nu_s,$$

$$\Delta_3\left(\frac{\nu_k}{k}\right) = -\frac{1}{k(k+1)(k+2)}\sum_{s=0}^{k-1}(s+1)(s+2)(s+3)\Delta_4\nu_s,$$

$$\Delta\lambda_k = \Delta\left(\frac{\mu_k}{k}\right) = \frac{1}{k(k+1)}\left(-\Delta_2\mu_0 + \sum_{s=1}^{k-1}\frac{1}{s(s+2)}\sum_{i=0}^{s-1}(i+1)(i+2)\Delta_3\nu_i\right), \quad (8)$$

$$\Delta_2\mu_0 = \frac{\alpha(\alpha-1)(n+2\alpha-3)}{2(\alpha+1)(\alpha+n)(\alpha+n-1)}$$

From the expressions for $\Delta_2\nu_k$, $\Delta_3\nu_k$, $\Delta_4\nu_k$ it follows that $\mathrm{sign}\Delta_2\nu_k = \mathrm{sign}(\alpha-1)$, $\mathrm{sign}\Delta_3\nu_k = \mathrm{sign}(\alpha-1)(\alpha-2)$, $\mathrm{sign}\Delta_4\nu_k = \mathrm{sign}(\alpha-1)(\alpha-2)(\alpha-3)$. Then we notice that λ_k is increasing for $\alpha \geq 2$ using (8), the representation for $\Delta_2\mu_0$ and that fact, that $\lambda_k > 0$ for $0 < \alpha < 1$ and $\lambda_k < 0$ for $\alpha \geq 2$.

To use Theorem B we need to have the estimates for $\Delta\lambda_k$ (more exactly for $\Delta\Lambda_k^{*(n)}$). Using the expressions for $\Delta_2\mu_0$ and $\Delta_3\nu_i$ we have (8):

$$\Delta\lambda_k = \frac{\alpha(\alpha-1)}{k(k+1)}\Bigg(-\frac{n+2\alpha-3}{2(n+1)(n+\alpha)(n+\alpha-1)} - \frac{(\alpha-2)c(k)}{(n+\alpha)(n+\alpha-1)(n+\alpha-2)} - \sum_{s=2}^{k-1}\sum_{i=1}^{s-1}\frac{1}{s(s+2)}\times \frac{(i+1)(i+2)(n-i+1)(n-i+2)\dots n}{(\alpha+n-i-2)(\alpha+n-i-1)\dots(\alpha+n)}\Bigg), \tag{9}$$

where $c(k) = \frac{3}{2} - \frac{2k+1}{k(k+1)}$.

Let $\alpha > 1$ and $1 \leq k \leq \left[\frac{n}{2}\right]$, here $[a]$ means the entire part of a. Change of order of summation in the double sum from (9), easy evaluations

$$\sum_{s=i+1}^{k-1}\frac{1}{s(s+2)} = \frac{1}{2}\left(\frac{2i+3}{(i+1)(i+2)} - \frac{2k+1}{k(k+1)}\right) \leq \frac{C}{i}$$

and some trivial estimations in (9) give us

$$|\Delta\lambda_k| \leq \frac{C}{k^2n^2} + \frac{C}{n^3}$$

and

$$|\Delta\Lambda_k^{(n)}| \leq C\max\left(\frac{1}{k^2}, \frac{1}{n}\right) \tag{10}$$

Let $\alpha > 1$ and $\left[\frac{n}{2}\right] < k \leq n$. To obtain the estimate for $\Delta\lambda_k$ like to (10) but for just mentioned α and k, another representation for $\Delta\lambda_k$ is needed. Taking into account the expression for λ_k and some easy identical transformations we have

$$\Delta\lambda_k = \frac{2k+1}{k^2(k+1)^2} - \frac{\alpha}{k(k+1)(n+1)} + \frac{(n-k+1)\dots n}{(\alpha+n-k)\dots(\alpha+n)}\times \frac{-\alpha k^2 - 2k\alpha - 2kn + 2k^2 - \alpha - n + k}{(k+1)^2k^2}$$

After some trivial estimations we obtain that $|\Delta\lambda_k| \leq \frac{C}{n^2}$ for our situation and

$$|\Delta\Lambda_k^{(n)}| \leq \frac{C}{n}. \tag{11}$$

So for $\alpha > 1$ and $1 \leq k \leq n$ we have the general estimation (see (10) and (11))

$$|\Delta\Lambda_k^{(n)}| \leq C\max\left(\frac{1}{k^2}, \frac{1}{n}\right). \tag{12}$$

Taking into account this estimate and using Theorem B, where A_k^α stands for a_k and A_k for $|\Delta A_k^{(n)}|$ (a series $\sum_{k=1}^{\infty} A_k$ is reduced here to a finite sum $\sum_{k=1}^{n} A_k$), we come to the conclusion that Lebesgue constants for $\|A\|$ (more exactly for $\|A\| - \alpha(1-\alpha)$) are bounded uniformly with respect to n. So the right side inequality in (7) is proved.

To prove the left-side inequality in (7) it remains only to prove the boundness of the Lebesgue constant for the transitional matrix. But using the comparison principle (Theorem A) for the left-side inequality in (7) one may notice that appropriate transitional matrix here is $\frac{1}{A^*}$, where A^* is the transitional matrix for the right-side inequality in (7).

First we will notice that if the elements of the matrix A^* satisfying (12) are separated from zero, i.e. there is $\delta < 0$ such that $A_k^{(n)} < \delta < 0$, then the elements of $\frac{1}{A^*}$ satisfies (12) as well. It remains to take into consideration the relation

$$\left|\Delta\left(\frac{1}{A_k}\right)\right| = \frac{1}{|A_k A_{k+1}|}\,|\Delta A_k|. \tag{13}$$

So to prove the left-side inequality in (7) it remains us only to prove that the elements of the matrix A are separated from zero. We will solve this problem in two stages.

Let $\alpha \geq 2$. For every fixed n the sequence of elements $\{A_k^{(n)}\}$ is increasing one. Then

$$A_k^{(n)} < A_n^{(n)} = \frac{(n+1)^2}{n^2}\left(1 - \frac{\alpha_n}{n+1} - \frac{A_0^\alpha}{A_n^\alpha}\right) \leq \frac{n+1}{n^2}(n+1-\alpha_n) =$$
$$= -\left(1+\frac{1}{n}\right)\left(\alpha - 1 - \frac{1}{n}\right) \leq \frac{3}{4} - \alpha < 0.$$

The proof of the same fact for $1 < \alpha < 2$ differs from the given one. The recurrence relation

$$A_{k+1}^{(n)} = \frac{(n-k)k^2}{(n-k+\alpha)(k+1)^2} A_k^{(n)} + \frac{\alpha(1-\alpha)(n+1)}{(k+1)(n-k+\alpha)}$$

is valid for $1 \leq k \leq n-1$ and may be checked directly. As $\frac{k^2}{(k+1)^2} \geq \frac{1}{4}$ so

$$\frac{(n-k)k^2}{(n-k+\alpha)(k+1)^2} \geq \frac{1}{4(1+\alpha)}.$$

Then

$$\frac{n+1}{(k+1)(n-k+\alpha)} \geq \frac{n+1}{n(1+\alpha)} \geq \frac{1}{1+\alpha},$$
$$\frac{\alpha(1-\alpha)(n+1)}{(k+1)(n-k+\alpha)} \leq \frac{\alpha(1-\alpha)}{1+\alpha}.$$

Taking into account that $A_k^{(n)} < 0$ in our situation we obtain

$$A_{k+1}^{(n)} \leq \frac{1}{4(1+\alpha)} A_k^{(n)} + \frac{\alpha(1-\alpha)}{1+\alpha} \leq \frac{\alpha(1-\alpha)}{1+\alpha} < 0$$

for $1 \le k \le n-1$.

Two estimations for $A_k^{(\alpha)}$ for $1 < \alpha < 2$ and for $\alpha \ge 2$ mean that $A_k^{(\alpha)}$ are separated from zero for $\alpha > 1$.

These observations, (13) and the boundedness of Lebesgue constants for the matrix A enable us to state that Lebesgue constants are bounded uniformly with respect to α for $\frac{1}{\Lambda}$ too. This automatically means that the left-side inequality in (7) holds.

Theorem 1 is proved.

Proof of Theorem 2. Theorem 2 follows immediately if we take into account Theorem 1 and the result (2) of II.K. Lebed and A.A. Avdienko from [1].

Theorems 1 and 2 are valid for $0 < \alpha \le 1$ too but the proofs in this case radically differs of given proofs.

Using Theorem 1 one may automatically prove the representations like to those in Theorem 2 for other different means of (1) involving arithmetic means as a special case. For different means (for example Riesz ones) the appropriate estimates may be given not only in terms of second moduli but in terms of higher order moduli (for given function and conjugated one if needed). Situation likes to be such that Cesaro means are worst in this sense.

Some constants C in our results are sharp. We would like to pay attention to a fact that the estimates in Theorems 1 and 2 are given not only from above as in [1], [2] or some other papers, but from below too. This a principal difference between our results and ones obtained by other authors.

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