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BOUNDEDNESS OF SOLUTIONS OF SOME SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper presents some mathematical methods for studying of specific class of analog processor networks. We present some theoretical results to the dynamic properties of dynamic feedback neural networks. The obtained results are in the form suitable for applications.

Hopfield [3], [12] has presented a type of neural network which is represented by the following systems of ordinary differential equations

$$x_i' = a_{ii}x_i + \sum_{j=1}^{n} b_{ij}g(x_j) + k_i, \tag{1}$$

where $i = 1, \ldots, n$ a_{ii} are negative, real constants, b_{ij} are positive, real constants and k_i are real constants. Nonlinear function g is bounded, monotonous and continuous. Papers [4], [6], [7], [9] present some results for function $g : \mathrm{R} \to \mathrm{R}$

$$g(x) = -\frac{1}{1 + e^{-Cx}}, \tag{2}$$

where C is positive real constant. It is shown that all solutions of the differential equation (1) are bounded, exists unique stable equilibrium point.

Papers [5], [10] examined system of nonlinear differential equations, with similar properties.

Consider nonlinear systems

$$\mathrm{x}' = \mathrm{C}\mathrm{x} + \mathrm{F}(\mathrm{x}), \tag{3}$$

where C is an n-by-n constant real matrix, $\mathrm{x} = (\mathrm{x}_1, \ldots, \mathrm{x}_n)^{\mathrm{T}}$, F is real function defined in some real n dimensional domain D in x space.

Paper [10] presented some possibility of the transformation of that system: If matrix C has m linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ corresponding to the real eigenvalues

$$\lambda_1, \ldots, \lambda_k$$

with multiplicities $n_1, \ldots, n_k$. Let X_j be the subspace of E_n generated by the system (3) for $j = 1, \ldots, k$. Let the matrix T be defined as follows:

$$\mathbf{T} = (\mathbf{v}_{11}, \mathbf{v}_{12}, \ldots, \mathbf{v}_{1n_1}, \mathbf{v}_{21}, \mathbf{v}_{22}, \ldots, \mathbf{v}_{2n_2},$$

$$\ldots, \mathbf{v}_{k_1}, \mathbf{v}_{k_2}, \ldots, \mathbf{v}_{kn_k}),$$

where $\mathbf{v}_{j1}, \ldots, \mathbf{v}_{jn_j}$ is any basis for X_j, $j = 1, \ldots, k$ and T is nonsingular matrix.

Transformation $\mathbf{x} = \mathbf{T}\mathbf{z}$ reduces the system $\mathbf{x}' = \mathbf{C}\mathbf{x} + \mathbf{F}(\mathbf{x})$ to the system

$$\mathbf{z}' = \mathbf{T}^{-1}\mathbf{A}\ \mathbf{T}\mathbf{z}\ + \mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{z}). \tag{4}$$

For distinct real eigenvalue a_{ii} of matrix C, can be written

$$x_i' = a_{ii}x_i + f_i(x_1, \ldots, x_n), \tag{5}$$

where $i = 1, \ldots, n$ and $\mathbf{T}^{-1}\mathbf{F}(\mathbf{T}\mathbf{z}) = \mathbf{f} = (f_1, \ldots, f_n)^T$.

Consider second system of the differential equations

$$y_i' = a_{ii}y_i + \bar{f}_i(y_1, \ldots, y_n), \tag{6}$$

where $i = 1, \ldots, n$.

This inequality will be applied in what follows.

Lemma 1 *Let u_i, h_i, φ_i, φ_i', ψ_i are continuous nonnegative functions, ψ_i' are continuous nonpositive functions on interval (t_0, ∞), let r_i are continuous positive increasing functions on interval $(0, \infty)$ and $i = 1, \ldots, m$. Then the inequalities defined for $u_i \geq 0$*

$$u_i(t) \leq \varphi_i(t) + \psi_i(t) \int\limits_{t_0}^{t} h_i(s) r_i[u_i(s)] ds, \tag{7}$$

implies the inequalities

$$u_i(t) \leq F^{-1}\left(\frac{1}{m}\sum_{j=1}^{m}[R_j(\varphi_j(t)) + \int_{t_0}^{t} \psi_j(s) h_j(s) ds]\right), \tag{8}$$

where $i = 1, \ldots, m$

$$R_j(\eta) = \int_{c_j}^{\eta} \frac{ds}{r_j(s)},$$

and $c_j = \varphi(t_0) > 0, \eta > 0$ *for* $i = 1, \ldots, m$,

$$\eta(t) = \frac{1}{m} \sum_{j=1}^{m} [R_j(\varphi_j(t)) + \int_{t_0}^{t} \psi_j(s) h_j(s) ds].$$

$$F(u) = \min_i \{R_i(u)\} \tag{9}$$

for $u > 0$. F^{-1} *is the inverse function of* F *such that* $\eta(t)$, *is within the domain of definition of* F^{-1}.

It is one of the generalization of known lemma of I. Bihari presented in [8].

Theorem 1 *Let functions* r_i, R_i *satisfy the conditions of Lemma 1,* a_{ii} *are negative constants, functions* $f_i, \bar{f}_i$ *are defined for* $-\infty < x_i < \infty, -\infty < y_i < \infty$, $i = 1, \ldots, n$ *and satisfying the bounds*

$$|f_i(t, x_1, \ldots, x_n) - \bar{f}_i(t, y_1, \ldots, y_n)| \leq h_i(t) r_i(|x_i - y_i|), \tag{10}$$

$\forall x_i \in \mathbb{R}, i = 1, \ldots, n$. *If there exist positive real constants* c_i, M_i, l_i *such that*

$$|R_i^{-1}(x)| < c_i + M_i |x|. \tag{11}$$

$$\lim_{t \to \infty} \int_{t_0}^{t} e^{a_{ii}(t-s)} h_i(s) ds \leq l_i. \tag{12}$$

Let $x_i(t)$, $y_i(t)$, $i = 1, \ldots, n$ *are solutions of the system (5) with initial conditions* $x_i(t_0) = x_i^0$ *and* $y_i(t)$, $i = 1, \ldots, n$ *are solutions of the system (6) with initial conditions* $y_i(t_0) = y_i^0$. *Then for all* $t \geq t_0 > 0$ *is* $|x_i(t) - y_i(t)|$, $i = 1, \ldots, n$ *bounded and*

$$\lim_{t \to +\infty} |x_i(t) - y_i(t)| \leq M_i l_i, \quad i = 1, \ldots, n.$$

Proof. From (5) and (6) follows that

$$[x_i(t) - y_i(t)] e^{-a_{ii} t} - [x_i(t_0) - y_i(t_0)] e^{-a_{ii} t_0} =$$

$$= \int_{t_0}^{t} e^{-a_{ii} s} \{ f_i[s, x_1(s), \ldots, x_n(s)] - \bar{f}_i[s, y_1(s), \ldots, y_n(s)] \} ds$$

Applying (9), initial conditions and properties of functions $r_i, i = 1, \ldots, n$ we have

$$|x_i(t) - y_i(t)| e^{-a_{ii} t} \leq |x_i^0 - y_i^0| e^{-a_{ii} t_0} + \int_{t_0}^{t} e^{-a_{ii} s} h_i(s) r_i(|x_i(s) - y_i(s)| e^{-a_{ii} s})\, ds.$$

Then from Lemma 1, for $m = 1$, we have

$$|x_i(t) - y_i(t)|e^{-a_{ii}t} \leq R_i^{-1}\Big\{R_i[|x_i^0 - y_i^0|e^{-a_{ii}t_0}] + \int_{t_0}^{t} e^{-a_{ii}s} h_i(s)\, ds\Big\}.$$

Using (11) we obtain

$$|x_i(t) - y_i(t)|e^{-a_{ii}t} \leq c_i + M_i R_i[|x_i^0 - y_i^0|e^{-a_{ii}t_0}] + M_i \int_{t_0}^{t} e^{-a_{ii}s} h_i(s)\, ds.$$

and from (12) we have

$$|x_i(t) - y_i(t)| \leq c_i e^{a_{ii}t} + M_i R_i[|x_i^0 - y_i^0|e^{-a_{ii}t_0}]e^{a_{ii}t} + M_i \int_{t_0}^{t} e^{a_{ii}(t-s)} h_i(s) ds \leq$$

$$\leq c_i e^{a_{ii}t} + M_i R_i[|x_i^0 - y_i^0|e^{-a_{ii}t_0}]e^{a_{ii}t} + M_i l_i.$$

$|x_i(t) - y_i(t)|$, $i = 1, \ldots, n$ are bounded with continuous functions and for $a_{ii} < 0, i = 1, \ldots, n$, $t > t_0 > 0$ we have

$$\lim_{t \to \infty} |x_i(t)| \leq M_i l_i.$$

The proof of the theorem is complete.

Paper [10] presented following theorem:

Theorem 2 *Let* $\forall(x_1, \ldots x_n) \in D$ *where* D *is closed, bounded set. Let* x_i, $i = 1, \ldots, n$ *are solutions of (5) with initial conditions* $x_i(t_0) = x_i^0 \in D$. *If exist real constants* b_{ij}, B_{ij}, k_i, K_i, *continuous functions* g, G *on closed, bounded set* D *and satisfying the bounds on* D

$$\sum_{j=1}^{n} b_{ij} g(x_j) + k_i \leq f_i(x_1, \ldots, x_n) \leq$$

$$\leq \sum_{j=1}^{n} B_{ij} G(x_j) + K_i. \qquad (13)$$

Then exist continuous functions φ_i, ψ_i *that inequalities*

$$\varphi_i(t) \leq x_i(t) \leq \psi_i(t) \qquad (14)$$

$i = 1, \ldots, n$ *hold for* $t > t_0 > 0$.

We can applicate theorem 1, theorem 2 to study of the Hopfield neuron networks, if some parameters of networks are not exactly.

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