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A DUAL ANALISYS OF A CONTACT PROBLEM WITH FRICTION

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1 INTRODUCTION

From the variational formulation of the contact problem with friction one obtine a variational inequation of the second kind containing a non-differentiable term. One of the most used method to remove the difficulties due to the non differentiable term is the dualisation method.

By choosing appropriate Langrangeanes one can transform the original problem of minimization, into a saddle point problem on a convex set of the form $K \times \Lambda$, where K denotes the restraints set and Λ is the Langrange's multipliers. This new formulation enable us to use certain known algorithms in order to compute the solution and to avoid the construction of complicated convex sets and to minimize some non-differentiable functionals.

It is important to stress that the Langrange's multipliers do have mechanical significance and may be directly approximated, in the framework of the dual problem.

The paper contains the following sections :

- existence of Langrange's multipliers;
- existence and unicity of the saddle point of the Langrangean

2 VARIATIONAL INEQUALITY ARISING FROM VARIATIONAL FORM OF THE CONTACT PROBLEM

The variational form of the contact problem with friction is the following variational inequality of the second kind:

Find $u \in V$ *such that*

$$b(u, v-u) + j_T(v) - j_T(u) \geq L(v-u), \qquad \forall v \in V \tag{1}$$

with $b(u,v) = a(u,v) + j_n(u,v)$, where:

- $V = \left\{ v \in (H^1(\Omega))^d : v = 0 \quad on \quad \Gamma_D \right\}$ *is the functions space*;
- $\Omega \subset \mathbb{R}^d$, $d = 2 \quad or \quad d = 3$ is an elastic body which occupies an open bounded Lipschitz domain , that come in contact with the rigid fundation.
- $\Gamma = \partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_C$ is the Lipschitz continuous boundary of the Ω,
- $a(u,v) = \int_\Omega \sigma(u)\varepsilon(v)dx$ is the virtual work produced by action of the stress σ on the strain ε ;
- $j_n(u,v) = \int_{\Gamma_C} c_n(u_n - g)_+^{m_n} v_n ds$ is the virtual work produced by normal pressure on the contact boundary Γ_C;
- $j_T(u) = \int_{\Gamma_C} c_n p_T \mid v_T \mid ds$ is the virtual work produced by tangential pressure on the contact boundary Γ_C; with $p_T = (u_n - g)_+^{m_T}$, $g \geq 0$ is the initial gap betwen Γ_C and the fundation, j_T is the non-differentiable functional;
- $L(v) = \int_\Omega F v dx + \int_{\Gamma_N} T v ds$ is the virtual work prodeced by the volume force F in Ω and by the surface traction T on Γ_N;
- c_n, c_T, m_n, m_T are material constants depending on boundary contact propieties, and $(\cdot)_+ = \max(\cdot, 0)$;

We denote the multiplier space:

$$\Lambda = \left\{ \lambda \in (L^2(\Gamma_C))^{d-1} : \mid \lambda \mid \leq 1 \text{ on } \operatorname{supp} p_T, \lambda = 0 \text{ on } \Gamma_C \setminus \operatorname{supp} p_T \right\} \tag{2}$$

and the restrains set:

$$K = \{ v \in V : v_n \leq g \}$$

The existence of a multiplies for the problem (1) is given by

Theorem 2.1. *The function* $u \in V$ *is the solution of the problem (1) if and only if the exists a* λ, *such that*

$$b(u,v) + \int_{\Gamma_C} c_T p_T \lambda v_T ds = L(v), \qquad \forall v \in V \tag{3}$$

$$\lambda \in \Lambda,\ \lambda u_T = | u_T |,\ on \quad \Gamma_C \tag{4}$$

Proof. Let $u \in V$ be the solution of the problem (1). By taking $v = 0$ and $v = 2u$ in (1), we obtain

$$b(u,v) + j_T(u) = L(u) \tag{5}$$

This, from (1) we have

$$b(u,v) + j_T(v) \geq L(v), \qquad \forall v \in V$$

Hence,

$$| L(v) - b(u,v) | \leq \int_{\Gamma_C} c_T p_T | v_T | ds, \qquad \forall v \in V$$

Therefore, then exists a continuous linear functional h on $H^{-1/2}(\Gamma_C)$, such that

$$L(v) - b(u,v) = h(v_T)$$
$$| h(v) | \leq c \| v_T \|_{H^{1/2}(\Gamma_C)}$$

By Hahn-Banach Theorem, we have the existence of a $\lambda \in L^{\infty}(\Gamma_C)$ a.i.

$$\| \lambda \|_{L^{\infty}(\Gamma_C)} \leq 1$$

and

$$b(v_T) = \int_{\Gamma_C} c_T p_T \lambda v_T ds, \qquad \forall v_T \in H^{1/2}(\Gamma_C)$$

i.e. we have relation (3).By taking v=u in (3) we obtain

$$b(u,u) + \int_{\Gamma_C} c_T p_T \lambda u_T ds = L(u)$$

which, combining with (5), gives

$$\int_{\Gamma_C} c_T p_T (\lambda u_T - |u_T|) ds = 0.$$

It then follows that $\lambda u_T = |u_T|$ on Γ_C. Conservely, let u, λ satisfying (3), (4). In (3), we replace v by v-u to obtain

$$b(u, v-u) + \int_{\Gamma_C} c_T p_T \lambda v_T ds - j_T(u) - L(v-u) = 0, \forall v \subset V \qquad (6)$$

Since

$$\int_{\Gamma_C} c_T p_T \lambda v_T ds \leq \int_{\Gamma_C} c_T p_T |v_T| ds$$

the relation (6) implies (1).Q.E.D.

3 VARIATIONAL INEQUALITY FORMULATION AS A SADDLE POINT PROBLEM

Following procedure from Céa and Glowinski [4], we define a Lagrangean L on $K \times \Lambda$.

Theorem 3.1 Let $u \in K, \lambda \in \Lambda$ satisfying (3) and (4). Then, (u, λ) is the unique saddle point of L on $K \times \Lambda$.

Proof. We first prove that if (u, λ) satisfy (3), (4), then (u, λ) is a saddle point of L on $K \times \Lambda$, i.e.

$$L(u, \eta) \leq L(u, \lambda) \leq L(v, \lambda), \forall v, \eta \in K \times \Lambda \qquad (7)$$

The first inequality in (7) holds trivially, for

$$\int_{\Gamma_C} c_T p_T \eta u_T ds \leq \int_{\Gamma_C} c_T p_T \mid v_T \mid ds = \int_{\Gamma_C} c_T p_T \lambda u_T ds \quad \forall \eta \in \Lambda$$

As for the second inequality in (7), we must show that

$$\frac{1}{2}b(u,u) - L(u) + \int_{\Gamma_C} c_T p_T \lambda u_T ds \leq \frac{1}{2}b(v,v) - L(v) + \int_{\Gamma_C} c_T p_T \lambda v_T ds \quad \forall v \in V \tag{8}$$

Since from (3),

$$\int_{\Gamma_C} c_T p_T \lambda (u_T - v_T) ds = L(u-v) - b(u, u-v)$$

it easily seen that (8) holds.

We have proved that (u, λ) is a saddle point of L on $K \times \Lambda$. Let us now prove the uniqueness of a saddle point of L. Assume (v_1, λ_1) is a saddle point of L. We need to show that $v_1 = u$, $\lambda_1 = \lambda$. From the first inequality in (7), we have

$$\int_{\Gamma_C} c_T p_T \eta u_T ds \leq \int_{\Gamma_C} c_T p_T \lambda u_T ds \quad \forall \eta \in \Lambda$$

In particular, taking $\mu = \lambda_1$, we obtain

$$\int_{\Gamma_C} c_T p_T (\lambda_1 - \lambda) u_T ds \leq 0 \tag{9}$$

Similarly, since (v_1, λ_1) is a saddle point of L, we have :

$$\int_{\Gamma_C} c_T p_T \eta v_{1T} ds \leq \int_{\Gamma_C} c_T p_T \lambda_1 v_{1T} ds \quad \forall \eta \in \Lambda$$

In particular, taking $\eta = \lambda$, we obtain

$$\int_{\Gamma_C} c_T p_T (\lambda - \lambda_1) v_{1T} ds \leq 0 \tag{10}$$

Adding (10) and (11), we get

$$\int_{\Gamma_C} c_T p_T (\lambda - \lambda_1)(u_T - v_{1T}) ds \leq 0 \tag{11}$$

From the second inequality in (7), we see that

$$b(u, v) - L(v) + \int_{\Gamma_C} c_T p_T \lambda v_T ds = 0 \quad \forall v \in K \tag{12}$$

Taking $v = v_1 - u$, we get

$$b(u, v_1 - u) - L(v_1 - u) + \int_{\Gamma_C} c_T p_T \lambda (v_{1T} - u_T) ds = 0 \tag{13}$$

Similar, we have

$$b(v_1, v) - L(v) + \int_{\Gamma_C} c_T p_T \lambda_1 v_T ds = 0 \quad \forall v \in K \tag{14}$$

and

$$b(v_1, v_1 - u) - L(v_1 - u) + \int_{\Gamma_C} c_T p_T \lambda_1 (v_{1T} - u_T) ds = 0 \tag{15}$$

Substracting (15) from (13), we_e obtain

$$b(v_1 - u, v_1 - u) = \int_{\Gamma_C} c_T p_T (\lambda_1 - \lambda)(u_T - v_{1T}) ds = 0$$

whichs by (11) is nonpositive. Hence, we must have

$$v_1 = u \tag{16}$$

Using (12), (14) and (11), we obtain

$$\int_{\Gamma_C} c_T p_T (\lambda_1 - \lambda) u_T ds = 0 \qquad \forall u_T \in H^{1/2}(\Gamma_C)$$

from which, we then have

$$\lambda_1 = \lambda$$

Q.E.D.

An consequence of Theorem 2.1 and 3.1 is

Proposition 3.1. For the solution u of the problem (1), there exists a unique $\lambda \in \Lambda$, with $\lambda v_T = | v_T |$, such that

$$b(u, v) = L(v) + \int_{\Gamma_C} c_T p_T \lambda v_T ds \quad \forall v \in K$$

Remarks.

The relationship between the solution of problem (1), denoted by $\bar{u}$, on a hand, and the saddle point (u, λ) of the Langrangean (8), by the other hand, is given by

$$\bar{u} = u \quad in \quad \Omega, \quad \sigma_T(\bar{u}) = -c_T p_T \lambda \quad a.e. \quad on \quad \Gamma_C \quad (17)$$

Therefore the contact problem with given friction can be approximated by solving the saddle point problem (7). This means the a minimization problem which contains a non-differentiable functional has been replaced by an other one in which the Lagrangean is regular with respect to each variable. From (17) we can deduce a very important mechanical interpretation for the Lagrange's multipliers.

One other benefit of this formulation consists in the possibility of exploiting of an Uzawa-type algorithm for the solution of the problem (7), which have to simultaneously compute the displacements as well as the contact tangential stresses.

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