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OPTIMAL MAINTENANCE POLICY OF A MACHINE

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Abstract. Optimal control methods have a lot of applications in economy, here in this paper are used for simultaneous optimization of maintenance policy and sale date of a machine. For linear variant of the associated optimal control problem we derive explicit expressions for optimal strategies.

1.Introduction

A machine is to be bought, used for productive purpose for some period, and then replaced. The machine is supposed to generate a constant rate of revenue over the capital invested in it.However, due to the physical depreciation and obsolescence phenomenon, the revenue rate decreases forcing the management to initiate a maintenance action in order to slow down the degradation of the machine's capability. The maintenance action here means money spent over and above the minimum spent on current repairs, and involves, preventive maintenance, reliability equipment, partial modernization, ergonomic improvements etc. The problem is to choose a maintenance schedule and replacement date so that to maximize the net revenue associated with the operation of the machine.

Using the optimal control methods, B.Naslund [6], G.L.Thompson [11],J.Y.Helmer [2], and I.Muntean [12], solved this problem under the assumption that maintenance costs are not so effective as to enhance the value of the machine over previous values.However, for a modification of the Thompson's linear model S.R.Arora and P.T.Lele [1], and D.N.Khandelwall, J.Sharma and L.M.Ray [5] have presented situations in which the salvage value of the machine may be increasing every where, or returning perodically to the purchased price, respectively.

Other machine replacement models in deterministic or stochastic setting are discussed in the papers of M.I.Kamien and N.L.Schwarts [4], S.P.Sethi [8], S.P. Tapiero and I.Venezia [10], S.P.Sethi and S.Chand [9], and D.G.Nguyen and D.N.P.Murthy [7].

In this paper we derive explicit expression for optimal stategies in linear variants, for a modification of Thompson's linear model introduced by S.R.Arora and P.T.Lele.

2.Mathematical model and necessary conditions optimality

The machine is purchased at time $t=0$ at price $x_0>0$, put into service at the same time, and replaced at time $t_1>0$. The residual value $x(t)$ of the machine at time $t\in[0,t_1]$ depends upon the folowing economic factors:

1°. In absence of maintenance action, the value $x(t)$ decreases with a quantity $a(t)\Delta t$ on the time interval $[t,t+\Delta t]$, where $a:[0,+\infty[\to\mathbb{R}$ is the obsolescence function;

2°. The depreciation rate $b(t)x(t)\Delta t$, which is due to change in physical characteristics and the performance of the machine

3°. The assignment of a sum of money $u(t)$ for maintenance action on $[t,t+\Delta t]$ produces an increase of $x(t)$ with a quantity $F(t,u(t))\Delta t$, where $u:[0,t_1]\to\mathbb{R}$ is a nonnegative function and $F:[0,+\infty[\times\mathbb{R}\to\mathbb{R}$ is a given function.

Taking into account the economic factors, the variation of the machine's value $\Delta x(t)=x(t+\Delta t)-x(t)$ is given by $\Delta x(t)=-a(t)\Delta t-b(t)x(t)\Delta t+F(t,u(t))\Delta t$, whence, assuming the differentiability of the function $x:[0,t_1]\to\mathbb{R}$, we obtain

$$\dot{x}(t)=-a(t)-b(t)x(t)+F(t,u(t))\,,\ x(0)=x_0$$

In establishing the net revenue we admit that the earnings $y(t)\Delta t$, due to the operation of the machine on the interval $[t,t+\Delta t]$ are proportional to the residual value x(t) and to the length of this interval, i.e., $y(t)\Delta t=rx(t)\Delta t$, where the constant r>0 is the product rate. The net revenue is the sum of the following three terms:

1. The negative of the initial price, i.e. $-x_0$;

2. The excess of earnings $y(t)\Delta t$ over the maintenance expenditure

$u(t)\Delta t$, which is to be discounted by a weight e^{-it}, and summed over all the operation period, i.e., $\int_0^{t_1}[r(t)-u(t)]e^{-it}dt$, where the positive constant i is the rate of interest, i<r;

3. The discounted residual value of the machine at the replacement date t_1, i.e., $x(t_1)e^{-it_1}$.

The net revenue now is given by:

$$-x_0+\int_0^{t_1}[rx(t)-u(t)]e^{-it}dt+x(t_1)e^{-it_1} \qquad (1)$$

Finally, let us denote by u_0 the upper bound of the maintenance costs, and suposse that $0<u_0\le+\infty$.

A triple $s=(t_1,u,x)$ is said to be a strategy associated with the above data, if $t_1>0$; $u:[0,t_1]\to\mathbb{R}$ is a piecewise continuous function(i.e., u is left-continuous and possesses a finite right-limit at each point in the open interval $]0,t_1[$, and it is continuous on the closed interval $[0,t_1]$ except for a finite subset E_u of $]0,t_1[$), such that for each t in $[0,t_1]$ we have $0\le u(t)\le u_0$, when $u_0<+\infty$, and $0\le u(t)$ when $u_0=+\infty$, respectively, and $x:[0,t_1]\to\mathbb{R}$ is a continuous function on $[0,t_1]$, which is differentiable on $[0,t_1]\setminus E_u$ and satisfies $x(0)=x_0$ and

$$\dot{x}(t)=-a(t)-b(t)x(t)+F(t,u(t)),\ for\ t\in[0,t_1]\setminus E_u \qquad (2)$$

The components u and x of a strategy are usually called **control** function and **state** function, respectively. Let us denote by S the set of all strategies and by $C:S\to\mathbb{R}$ the cost functional defined as follows:

$$C(s) = \int_0^{t_1} L(t, u(t), x(t))\,dt \qquad for\ s = (t_1, u, x) \in S,$$

where $L: [0, \infty[\times \mathbb{R}^2 \to \mathbb{R}$ is the function given by

$$L(t, u, x) = -[(r - i - b(t))x - u - a(t) + F(t, u)]\,e^{-rt}$$

Recalling (2), the negative of the net revenue in (1) associated with a strategy $s = (t_1, u, x)$ can be rewritten as

$$\int_0^{t_1} -[rx(t) - u(t)]\,e^{-rt}\,dt - [x(t_1)e^{-rt_1} - x(0)] =$$

$$= \int_0^{t_1} -[rx(t) - u(t)]\,e^{-rt}\,dt - \int_0^{t_1} -\frac{d}{dt}[x(t)\,e^{-rt}]\,dt = \int_0^{t_1} L(t, u(t), x(t))\,dt = C(s)$$

Now the maximization of the net revenue is equivalent to the problem: find a strategy $s_* = (t_*, u_*, x_*)$ in S such that $C(s_*) \le C(s)$ for every strategy $s = (t_1, u, x)$ in S.

This is an optimal control problem with integral cost, free final time and free final state. We derive necessary conditions of optimality from the well-known maximum principle of Pontrjagin. To this end suppose that the functions a, b and F are continuous together with their partial derivatives of first order, and introduce the Hamiltonian of the problem

$$H(t, u, x, p) = [(r - i - b(t))x - u - a(t) + F(t, u)]\,e^{-rt} + p[-a(t) - b(t)x(t) - F(t, u)]$$

Now suppose that $s_* = (t_*, u_*, x_*) \in S$ is an optimal strategy, i.e., it is a solution of the above optimal control problem. Then, by the maximum principle of Pontrjagin, there exists a continuous function $p: [0, t_*] \to \mathbb{R}$, which is differentiable on $[0, t_*] \setminus E_{u_*}$, such that the differential equation

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(t, u_*(t), x_*(t), p(t)) \qquad (3)$$

is satisfied for $t \in [0, t_*] \setminus E_{u_*}$, the maximum principle

$$H(t, u, x_*(t), p(t)) \leq H(t, u_*(t), x_*(t), p(t)), \quad t \in [0, t_*] \setminus E_{u_*} \qquad (4)$$

holds for $u \in [0, u_0]$ when $u_0 < +\infty$, and for $u \geq 0$ when $u_0 = +\infty$, and the transversality conditions:

$$p(t_*) = 0 \text{ and } H(t_*, u_*(t_*), x_*(t_*), p(t_*)) = 0 \qquad (5)$$

are fulfilled.

Now we suppose that function b is a constant function, i.e., $b(t) = b$ for every $t \in [0, t_*]$ and since p is continuous, the integration of (3) with the final condition $p(t_*) = 0$ in (5) leads to

$$p(t) = \frac{r - i - b}{b + i}\left[e^{-(i+b)t} - e^{-(i+b)t_*}\right]e^{bt}, \quad t \in [0, t_*],$$

so that the value of the Hamiltonian in the left side (4) can be written as:

$$H(t, u, x_*(t), p(t)) = \varphi(t) + h(t, u)$$

where

$$\varphi(t) = (r - i - b)x_*(t)e^{-it} - bp(t)x(t) - a(t)\left[e^{-it} + p(t)\right]$$

$$h(t, u) = e^{-it}\left[-u + g(t)F(t, u)\right]$$

and

$$g(t) = 1 + \frac{r - i - b}{b + i}\left[1 - e^{(i+b)(t - t_*)}\right]$$

We suppose now that $b + i < r$ and it is clearly that $g(t) > 0$ for $t \in [0, t_*]$.

Now the maximum principle (4) becomes

$$h(t, u) \leq h(t, u_*(t)), \quad t \in [0, t_*] \setminus E_{u_*} \qquad (6)$$

for $u \in [0, u_0]$ when $u_0 < +\infty$, and for $u \geq 0$ when $u_0 = +\infty$, and the equalities in (5) imply

$$(r - i - b)x_*(t_*) - u_*(t_*) - a(t_*) + F(t_*, u_*(t_*)) = 0 \qquad (7)$$

From (6) we see that, for each t in $[0, t_*] \setminus E_{u_*}$ the value $u_*(t)$ of the optimal control is a maximum point for the function $u \to h(t, u)$ on the interval

$[0,u_0]$ when $u_0<+\infty$, and on the interval $[0,+\infty[$ when $u_0=+\infty$. Hence it will be useful to search the sign of the partial derivative

$$\frac{\partial h}{\partial u}(t,u)=\frac{e^{\,u}}{g(t)}\left[-G(t)+\frac{\partial F}{\partial u}(t,u)\right],\ t\in[0,t_*]\setminus E_*, \tag{8}$$

near the point $u=u_*(t)$. Here the function $G:[0,t_*]\to R$, given by

$$G(t)=\frac{1}{g(t)} \tag{9}$$

is positive continuous and strictly increasing since

$$\frac{dG(t)}{dt}=\frac{-g'(t)}{[g(t)]^2}>0 \text{ for every } t\in[0,t_*].$$

Relations (6) and (8) facilitate the study of the optimal maintenance policy u_* on the length of the time interval $[0,t_*]$, and equation (7) permits the computation of the replacement date t_*.

The optimal strategies will be investigated for the linear variant of the optimal control problem.

3.Linear variant

Admit that the function $(t,u)\to F(t,u)$ is linear with respect to the second variable, i.e., F has the form $F(t,u)=f(t)u$, where the function $f:[0,+\infty[\to R$ is continuous together with its first derivative. In this case the differential equation (2) takes on the form

$$\dot{x}(t)=a(t)-bx(t)+f(t)u$$

$$x(0)=x_0$$

3.1.THEOREM. Suppose $u_0<+\infty$, f is decreasing on the interval $[0,+\infty[$ and $v_*=(t_*,u_*,x_*)$ is an optimal strategy. Then the optimal control u_* can be determined as follows:

a) If $f(t) < G(t)$ for all $t \in [0, t_*]$, we have $u_*(t) = 0$ when $t \in [0, t_*]$;

b) If $f(t) > G(t)$ for all $t \in [0, t_*]$, we have $u_*(t) = u_0$ when $t \in [0, t_*]$;

c) If neither of the conditions in a) and b) are satisfied, we have

$$u_*(t) = \begin{cases} u_0, & when \quad t \in [0, t_0], \\ 0, & when \quad t \in]t_0, t_*], \end{cases}$$

where t_0 is the single root in the open interval $]0, t_*[$ of the equation $f(t) = G(t)$.

Proof. If the condition in a) is satisfied, from (8) we get

$$\frac{\partial h}{\partial u}(t,u) = \frac{e^{-it}}{g(t)}[-G(t) + f(t)] < 0$$

for all $t \in]0, t_*] \setminus E_{u_*}$ and all $u \in [0, u_0]$. Therefore, the function $u \mapsto h(t, u)$ is strictly decreasing on the interval $[0, u_0]$, so that it attains its maximun value at the point $u=0$. Hence $u_*(t) = 0$ when $t \in [0, t_*] \setminus E_{u_*}$. Since E_{u_*} is a finite subset of $]0, t_*[$ and u_* is a continuous function at the point $t=0$ and left continuous on the interval $]0, t_*[$, it follows that $u_*(t) = 0$ for every $t \in [0, t_*]$.

In the condition in b) is satisfied, then the function $u \mapsto h(t, u)$ is strictly increasing on $[0, u_0]$, for each $t \in]0, t_*] \setminus E_{u_*}$, hence this function attains its maximum value at the point $u = u_0$. As above, we have $u_*(t) = u_0$ for each $t \in [0, t_*]$.

If the condition in c) is fulfilled, there exist a point t' in $]0, t_*]$ and a point t'' in $[0, t_*[$ such that $f(t') > G(t')$ and $f(t'') \leq G(t'')$.

Since the function $t \mapsto G(t) - f(t)$ is strictly decreasing and continuous on $[0, t_*]$, the inequality $t' \leq t''$ holds, and there exists a single t_0 in

$[t', t'']$ such that $f(t_0) = G(t_0)$. Consequently, $f(t) > G(t)$ for $0 \le t < t_0$, and $f(t) < G(t)$ for $t_0 < t \le t''$. Now, we take into account (8) to complete the proof of the theorem.

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