Buletinul Ştiinţific al Universităţii din Baia Mare
Seria B, Matematică-Informatică, vol.XIII(1997),139-146

ON FIXED POINTS FOR SURJECTIVE MAPPINGS SEMI-HAUSDORFF SPACES

Valeriu POPA

Abstract. In the present paper two fixed point theorems for surjective mappings in semi-Hausdorff spaces have been established.

Key words and phrases: surjective mapping, fixed point, semi-Hasudorff space.
1991 AMS Mathematica Subject Classification: 54H25, 47H10

A number of author have discussed contractive type mappings on a metric space which are generalizations or well-known result of Edelstein [3].

Constantin [1] extend this result for a pair of mappings of a noncomplete metric space. Fisher and Ray [4], Popa [5] and others extends the result of Edelstein for mappings on a Hausdorff space. Recently, Dubey [2] extend the result from [5] for mappings on a semi-Hausdorff space.

It is known that a topological space X is said to be semi-Hausdorff if and only if every sequence in X has at most one limit. Every Hausdorff space is semi-Hausdorff but not conversely. Every semi-Hausdorff space is T_1 but not conversely [2].

Let $\mathbb{R}_+$ denote the set of all non-negative reals number and $\mathbb{N}$ the set of all positive integers. We denote $\mathcal{G}$ the set of all function $g:\mathbb{R}_+^3 \to \mathbb{R}_+$ with property: if $u, v \geq 0$ are such

$$(G_1) \quad u > g(v, v, u) \quad or$$

$$(G_2) \quad u > g(v, u, v),$$

then $u > v$.

Ex.1. $g(t_1,t_2,t_3) = [a t_1^2 + b t_1 \cdot t_2 + c t_3 \cdot t_1 + d t_2 t_3]^{1/2}$, where $a > 1$, $b, c, d \geq 0$. Let $u > g(v,v,u)$ be, then we have

$$u^2 > a v^2 + b v^2 + c u v + d u v \,; \quad u^2 - (c+d) u v - (a+b) v^2 > 0$$

If $v = 0$, then $u > v = 0$. If $v \neq 0$, then $f(t) = t^2 - (c+d)t - (a+b) > 0$, where $t = \dfrac{u}{v}$. As $f(1) = 1 - (a+b+c+d) < 0$, let $k > 1$ be the root of equation $f(t) = 0$. Then $f(t) > 0$ for $t > k$ and thus $u > k \cdot v > v$.

Similarly, $u > g(v,u,v)$ implies $u > v$

Ex.2. $g(t_1,t_2,t_3) = \left[a t_1^k + b t_2^k + c t_3^k\right]^{1/k}$, where $k > 1$, $0 \leq b, c < 1$ *and* $a > 1$. Let $u > g(v,v,u)$ be, then we have

$u^k > a v^k + b v^k + c u^k$; $u^k(1-c) > (a+b) v^k$; $u > \left(\dfrac{a+c}{1-b}\right)^{1/k} \cdot v > v$ because $\dfrac{a+b}{1-c} > 1$.

Similarly, $u > g(v,u,v)$ implies $u > \left(\dfrac{a+b}{1-c}\right)^{1/k} \cdot v > v$ because $\dfrac{a+c}{1-b} > 1$.

The purpose of this paper is to prove two theorems for surjective mappings in semi-Hausdorff spaces.

THEOREM 1. Let T be a surjective continuous mapping of semi-Hausdorff spaces X into itself and let f be a symmetric continuous mapping of $X \times X$ into non-negative reals such that

a) $f(x,y) \neq 0$, $\forall\ x \neq y \in X$,

b) There is $g \in \mathcal{G}$ such that

(1.1) $f(Tx,Ty) > g(f(x,y), f(x,Tx), f(y,Ty))$ for all $x \neq y \in X$;

c) The inequality

(1.2) $g(f(x,y), f(x,x), f(y,y)) > f(x,y)$, holds for all $x \neq y \in X$.

If for some $x_0 \in X$, the sequence $\{x_n\}$, where $x_n \in T^{-1}x_{n-1}$, $n = 1,2,\ldots$ has a convergent subsequence, then T has a unique fixed point.

Proof. Let $x_0 \in X$ be. Since f is surjective, there exists an element $x_1 \in X$ satisfying $x_1 \in T^{-1}x_0$. In the same way we can take $x_n \in T^{-1}x_{n-1}$, $n = 1,2,\ldots$ If $x_m = x_{m-1}$ for some $m \in \mathbb{N}$, then x_m is a fixed point of T. Without loos generality, we can suppose $x_{n-1} \neq x_n$ for every $n = 1,2,\ldots$. Then from (1.1) we have

$$f(x_{n-1},x_n) = f(Tx_n,Tx_{n+1}) > g((fx_n,x_{n-1}), f(x_n,Tx_n), f(x_{n+1},Tx_{n+1})) =$$

$$= g(f(x_n,x_{n-1}), f(x_n,x_{n-1}), f(x_{n-1},x_n)).$$

By (G_2) we obtain $f(x_{n-1},x_n) > f(x_n,x_{n-1})$ and thus the sequence $\{f(x_n,x_{n-1})\}$ is convergent to u.

Again $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to some $x \in X$. From continuity of T, we have

$$Tx = T\lim x_{n_k} = \lim Tx_{n_k} = \lim x_{n_k-1},$$

$$T^2x = T(Tx) = T(\lim x_{n_k-1}) = \lim Tx_{n_k-1} = \lim x_{n_k-2},$$

By continuity of f we have

$$f(x,Tx)=f(\lim x_{n_k},\lim x_{n_k-1})=\lim f(x_{n_k},x_{n_k-1})=u=\lim f(x_{n_k-1},x_{n_k-2})=$$

$$=f(\lim x_{n_k-1},\lim x_{n_k-2})=f(Tx,T^2x).$$

Thus

(1.3) $$f(x,Tx)=f(Tx,T^2x).$$

Now we prove that x is fixed point of T. If $x\neq Tx$ we would obtain from (1.1)

$$f(Tx,T^2x)>g(f(x,Tx),f(x,Tx),f(Tx,T^2x)).$$

By (G_1) we have $f(Tx,T^2x)>f(x,Tx)$ a contradiction of (1.3).

To prove the uniqueness, let $y\neq x$ be another fixed point of T. Then by (1.1) and (1.2) we have

$$f(x,y)=f(Tx,Ty)>g(f(x,y),f(x,Tx),f(y,Ty))=$$

$$=g(f(x,y),(x,x),(y,y))>f(x,y),$$

a contradiction.

COROLLARY 1. Let T be a continuous surjective mapping of a metric space (X,d) into itself such that

a) $$d^2(Tx,Ty)>ad^2(x,y)+bd(x,y)d(x,Tx)+$$
$$+cd(x,y)d(y,Ty)+d\cdot d(x,Tx)\cdot d(y,Ty),$$

where $a>1$; $b,c,d>0$, or

b) $$d^k(Tx,Ty)>ad^k(x,y)+bd^k(x,Tx)+cd^k(y,Ty),$$

where $k\geq 1$, $a>1$, $0\leq b,c<1$, for every $x\neq y\in X$.

If for some $x_0 \in X$, the sequence $\{x_n\}$, where $x_n \in T^{-1}x_{n-1}$, $n = 1, 2, \ldots$ has a convergent subsequence, then T has a unique fixed point.

Proof. Follows from Ex 1 and 2 and Theorem 1.

THEOREM 2. Let S and T be continuous surjective mappings on a semi-Hausdorff space X into itself and let f be a symmetric continuous mapping of $X \times X$ into non-negative reals such that

a) $f(x, y) \neq 0$, $\forall\ x \neq y \in X$;

b) There is $g \in \mathcal{G}$ such that

$$f(Sx, Ty) > g(f(x, y), f(x, Sx), f(y, Ty)) \tag{2.1}$$

for all $(x, y) \in X \times X - \{(x, x) : x \in X \ \ \textit{and} \ \ Sx = Tx\}$;

c) The inequality

$$g(f(x, y), f(x, x), f(y, y)) > f(x, y) \tag{2.2}$$

holds for all $x \neq y \in X$.

If for some $x_0 \in X$, the sequence $\{x_n\}$, where $x_{2n+1} \in S^{-1}x_{2n}$ and $x_{2n+2} \in T^{-1}x_{2n+1}$ for $n = 0, 1, 2, \ldots$ has a convergent subsequence, then either S or T has a fixed point z. Further, if z is a common fixed point of S and T, then z is the unique common fixed point of S and T.

Proof. Let $x_0 \in X$ be. Since S is surjective there is a point $x_1 \in S^{-1}x_0$. Since T is surjective there is a point $x_2 \in T^{-1}x_1$. Continuing in that manner one obtains a sequence $\{x_n\}$ with $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$. If there exists a $n \in \mathbb{N}$ such that $x_{2n+1} = x_{2n}$ *or* $x_{2n+1} = x_{2n+2}$ then $x_{2n+1} = x_{2n} = Sx_{2n+1}$ and x_{2n+1} is a fixed point of S or $x_{2n+1} = x_{2n+2} = Tx_{2n+2}$ and x_{2n+2} is a fixed point of T. Without loos generality, we can suppose that $x_n \neq x_{n+1}$ for every n. Then we have by (2.1)

$$f(x_{2n}, x_{2n+1}) = f(Sx_{2n-1}, Tx_{2n+2}) \geq g(f(x_{2n+1}, x_{2n+2}), f(x_{2n+1}, Sx_{2n+1}), f(x_{2n+2}, Tx_{2n+2})) =$$
$$= g(f(x_{2n+1}, x_{2n+2}), f(x_{2n+1}, x_{2n}), f(x_{2n+2}, x_{2n+1})).$$

By (G_2) we have $f(x_{2n}, x_{2n+1}) > f(x_{2n+1}, x_{2n+2})$.
On the other hand we have by (2.1)

$$f(x_{2n+1}, x_{2n+2}) =$$
$$= f(Sx_{2n+3}, Tx_{2n+2}) \geq g(f(x_{2n+2}, x_{2n+3}), f(x_{2n+3}, Sx_{2n+3}), f(x_{2n+2}, Tx_{2n+2})) =$$
$$= g(f(x_{2n+2}, x_{2n+3}), f(x_{2n+3}, x_{2n+2}), f(x_{2n+1}, x_{2n+2}))$$

By (G_1) we have $f(x_{2n+1}, x_{2n+2}) > f(x_{2n+2}, x_{2n+3})$ and thus the sequence $\{f(x_n, x_{n+1})\}$ is convergent to u.

Again $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ convergent to some x. Now, we choose a subsequence $\{x_{n_{k_i}}\}$ such that $\{n_{k_i}\}'s$ are all even as all odd. Let all $\{n_{k_i}\}'s$ be even. Also $\{x_{n_{k_i}}\}$ is convergent to x. By continuity of S and T we have

$$Tx = T(\lim x_{n_{k_i}}) = \lim Tx_{n_{k_i}} = \lim x_{n_{k_i}+1}$$

$$STx = S(\lim (x_{n_{k_i}+1})) = \lim S(x_{n_{k_i}+1}) = \lim x_{n_{k_i}+2}.$$

By continuity of f we have

$$f(x,Tx)=f(\lim x_{n_k},\lim x_{n_k+1})=\lim f(x_{n_k},x_{n_k+1})=u=\lim f(x_{n_k-1},x_{n_k-2})=$$

$$=f(\lim x_{n_k-1},\lim x_{n_k-2})=f(Tx,STx).$$

Thus

$$(2.3)\qquad f(x,Tx)=f(Tx,STx)$$

Let suppose that $Tx\neq x$. Then we have by (2.1)

$$f(Tx,STx)=f(STx,Tx)>g(f(Tx,x),f(Tx,STx),f(x,Tx)).$$

By (G_2) we have that $f(Tx,STx)>f(Tx,x)$ a contradiction of (2.3).

Similarly we can show that if all $\{n_k\}$'s are odd then $x=Sx$.

If $x=Sx=Tx$ we have that there is no other point $y\neq x$ such that $Sy=Ty=y$. Since otherwise we would have by (2.1) and (2.2)

$$f(x,y)=f(Sx,Ty)>g(f(x,y),f(x,Sx),f(y,Ty))=$$

$$=g(f(x,y),f(x,x),f(y,y))>f(x,y),$$

a contradiction.

COROLLARY 2. Let S and T be two continuous surjective mappings of a metric space (X,d) into itself such that

a) $$d^2(Sx,Ty)>ad^2(x,y)+bd(x,y)d(x,Sx)+$$
$$+cd(x,y)d(y,Ty)+d\cdot d(x,Sx)d(y,Ty)$$

where $a > 1$, $b, c > 0$, *or*

b) $$d^k(Sx,Ty) \geq a \cdot d^k(x,y) + bd^k(x,Sx) + cd^k(y,Ty)$$

where $a > 1$, $0 \leq b, c < 1$, for all

$$(x,y) \in X \times X - \{(x,x) : x \in X \ \ and \ \ Sx = Tx\}$$

If for some $x_0 \in X$, the sequence $\{x_n\}$, where $x_{2n+1} \in S^{-1}x_{2n}$ and $x_{2n+2} \in T^{-1}x_{2n+1}$, $n = 0,1,2,...$ has a convergent subsequence, then either S or T has a fixed point z. Further, if z is a common fixed point of S and T, then z is the unique common fixed point of S and T.

Proof. Follow from Ex 1 and 2 and Theorems 2.

REFERENCES

1. **A.,CONSTANTIN,** On fixed points in noncomplete metric spaces, Publ. Math., Debrecen, 40(1992), 297-303
2. **B.,N.,DUBEY,** Some fixed points theorems in semi-Hausdorf spaces, Pure and Appl.Math.Sci., 39(1994), 165-170
3. **M.,EDELSTEIN,** On fixed and periodic points under contractive mappings, J.London Math.Soc., 37(1962), 74-79
4. **B.,FISHER and B.,K.,RAY,** On fixed points for a class of mappings, Iraqi J.Sci., 22(1981), 255-259
5. **V.,POPA,** Some unique fixed point theorems in Hausdorff spaces, Indian J.Pure Appl.Math., 14(8)(1983), 713-717

Received 26.05.1997

Department of Mathematics-Physics
University of Bacău
5500 Bacău
ROMANIA