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# COLLOCATOIN METHOD OF SOLVING NONLINEAR SINGULAR INTEGRAL EQUATIONS GIVEN ON CLOSED SMOOTH CONTOUR

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Let $\Gamma$ be a closed smooth contour [1, p.14] bounding a simple connected region $F^+$ of the complex plane $C$ containing the point $z = 0$. In the Banach space of functions $H_\beta(\Gamma)$ [1, c.173] satisfying on $\Gamma$ the Hölder condition with the exponent $\beta (0 < \beta < 1)$ consider a nonlinear singular integral equation (SIE) of the form

$$A(\varphi) \equiv \Phi[t; \varphi(t); S_\tau h(t, \tau; \varphi(\tau))] = f(t), \tag{1}$$

where $\Phi[t; u; v](t \in \Gamma; |u|, |v| < \infty), h(t, \tau; u)(t, \tau \in \Gamma, |u| < \infty)$ and $f(t)$ are known continuous functions of their arguments, the singular integral

$$S_\tau h(t, \tau; \varphi(\tau)) \equiv \frac{1}{\pi i} \int_\Gamma \frac{h(t, \tau; \varphi(\tau))}{\tau - t} d\tau, t \in \Gamma,$$

is understood in the meaning of Cauchy principal value, and $\varphi(t) (t \in \Gamma)$ is an unknoun function.

In this work we propose a computing scheme of collocation method for the equations (1) and using the results of [2, p.75] give a theoretical foundation of this scheme in the Hölder spaces. Note that earlier in the paper of the author [3] the foundation of collocation method for the equation (1) is obtained for the case of Lyapunov's contour $\Gamma$ [1, p.14]. In present work the following results are obtained: 1) the class of contours $\Gamma$ is essentally extended; 2) the basis of method is carried out for the case, when the searching solution $\varphi(t)$ of the equation (1) belongs to the space $H^r_\alpha(\Gamma), r = 0, 1, ...,$ that is the function $\varphi(t)$ is $r$- times differentiable and $\varphi^{(r)}(t) \in H_\alpha(\Gamma)$; 3) the conditions on functions $\Phi[t; u; v]$ and $h(t, \tau; u)$ and conditions of reversing of the operator $A^{'}(\varphi)$ in $H_{\beta}(\Gamma)$ are defined more precisely; 4) the estimation of convergence rate of approximative solution in Hölder's spaces ($ln(n)$ instead of $ln^{4}(n)$ ) is essentially improved; 5).the proof of convergence is simplified.

We seek for the approximate solution of nonlinear SIE (1) as a polinomial

$$\varphi_n(t) = \sum_{k=-n}^{n} \alpha_k t^k \quad (t \in \Gamma), \tag{2}$$

the unknown coefficients of which $\{\alpha_k\}_{k=-n}^{n}$ will be determinated from the system of nonlinear equations (SNE)

$$\Phi[t_j; \varphi_n(t_j); \frac{1}{\pi i} \int_{\Gamma} \frac{h(t_j, \tau; \varphi_n(\tau))}{\tau - t_j} d\tau] = f(t_j), j = \overline{0, 2n}, \tag{3}$$

where $\{t_j\}_{j=0}^{2n}$ is a set of pairwize distinct points on $\Gamma$.

**Theorem.** *Let the following conditions be fulfilled:*

1) *the functions* $\Phi[t; u; v] \in H_{\alpha,1,1}(\Gamma)$ *and* $h(t, \tau; u) \in H_{\mu;\alpha;1}(\Gamma)(0 < \alpha \leq \mu \leq 1)$ *satisfy the conditions*

$$\Phi^{''}_{uu}[t; u; v], \Phi^{''}_{vv}[t; u; v], \Phi^{''}_{uv}[t; u; v] \in H_{\alpha;1,1}(\Gamma), h^{''}_{uu}(t, \tau; u) \in H_{\mu;\alpha;1}(\Gamma);$$

2) *the nonlinear SIE (1) in some sphere of the space* $H_{\beta}(\Gamma)(0 < \beta < \alpha \leq 1)$ *has a unique solution* $\varphi(t) \in H^{r}_{\alpha}(\Gamma), r = 0, 1, ...;$

3) $C^{2}_{\varphi}(t) - D^{2}_{\varphi}(t) \neq 0(t \in \Gamma)$, $ind[(C_{\varphi}(t) + D_{\varphi}(t)) \cdot (C_{\varphi}(t) - D_{\varphi}(t))^{-1}] = 0(t \in \Gamma)$, *where*

$$C_{\varphi}(t) = \Phi^{'}_{u}[t; \varphi(t); S_{\tau} h(t, \tau; \varphi(\tau))] \in H^{r}_{\alpha;1;1}(\Gamma),$$

$$D_{\varphi}(t) = \Phi^{'}_{v}[t; \varphi(t); S_{\tau} h(t, \tau; \varphi(\tau))] \cdot h^{'}_{u}(t, t; \varphi(t)) \in H^{r}_{\alpha;1;1}(\Gamma);$$

4) $dim Kern A^{'}(\varphi) = 0$ , *where* $A^{'}$ *is defined in the following way:* $\forall \varphi^{o}(t) \in H_{\beta}(\Gamma)$,

$$(A^{'}(\varphi^{o})g)(t) \equiv \Phi^{'}_{u}[t; \varphi^{o}(t); S_{\tau} h(t, \tau; \varphi^{o}(\tau))]g(t) +$$

$$+ \Phi^{'}_{v}[t; \varphi^{o}(t); S_{\tau} h(t, \tau; \varphi^{o}(\tau))] \cdot S_{\tau}[h^{'}_{u}(t, \tau; \varphi^{o}(\tau))g(\tau)] =$$

$$= C_{\varphi^{o}}(t)g(t) + D_{\varphi^{o}}(t)S_{\tau}g(\tau) +$$

$$+ \frac{1}{\pi i} \int_{\Gamma} \Phi^{'}_{v}[t; \varphi^{o}(t); S_{\tau} h(t, \tau; \varphi^{o}(\tau))] \frac{h^{'}_{u}(t, \tau; \varphi^{o}(\tau)) - h^{'}_{u}(t, t; \varphi^{o}(\tau))}{\tau - t} g(\tau) d\tau;$$

*5) the points $t_j (j = \overline{0, 2n})$ form on $\Gamma$ a system of Fejer's nodes [4, p.36], i.e.*

$$t_j = \psi(w_j), w_j = exp\left(\frac{2\pi i}{2n+1}(j-n)\right), i^2 = -1, j = \overline{0, 2n},$$

*$t = \psi(w)$ is the Riemann's function which realizis the conform mapping of exterior of the unitary circumfirence $\Gamma_o (= |W| = 1)$ on the exterior of $F^+ \cup \Gamma$ such that $\psi(\infty) = \infty, \psi'(\infty) = const > 0$.*

*Then there exists a 2n+1 dimensional point $\{y_k\}_{k=-n}^{n}$, in a neighborhood of which SNE (3) has a unique solution $\{\alpha_k\}_{k=-n}^{n}$ for all n begining with a certain one. The approximate solutions (2) converge as $n \to \infty$ in the norm of $H_\beta(\Gamma)$ to $\varphi(t)$ for any function $f(t) \in H_\alpha^r(\Gamma)$. For the rate of convergence the following estimation holds*

$$\| \varphi - \varphi_n \|_\beta = O(n^{-r+\beta-\sigma(\alpha)} \cdot ln(n)) H(\varphi^{(r)}; \sigma(\alpha)), \qquad (4)$$

*where $\sigma(\alpha) = \alpha$, if $0 < \alpha < 1$ and $\sigma(\alpha) = \alpha - \varepsilon$ if $\alpha = 1$.*

**Proof.** By conditions 1) of the theorem the nonlinear operator $A$, determined by the left-hand side of the equatuon (1) is Freshet differentiable [5] at every point $\varphi^0$ of $H_\beta(\Gamma)(0 < \beta < \alpha \leq 1)$ and its derivative has the form described by condition 4) of the theorem. Moreover, in the sphere $\| g - \varphi^o \|_\beta \leq r (r > 0)$ of $H_\beta(\Gamma)$, the linear operator $A'$ satisfies the Lipshitz condition

$$\| A'(g_1) - A'(g_2) \|_\beta \leq L \cdot \| g_1 - g_2 \|_\beta, \qquad (5)$$

where $L$ is a completely definite constant, its value depends on $r$, on element $\varphi^o(t) \in H_\beta(\Gamma)$ and on functions $\Phi$ and $h$.

Let $\chi(t)$ be an arbitrary continuous on $\Gamma$ function and

$$U_n(\chi) = U_n(\chi, t) = \sum_{j=0}^{2n} \chi(t_j) l_j(t)$$

its interpolating Lagrange polynomial, constructed by nodes $\{t_j\}_{j=0}^{2n} \subset \Gamma$, where

$$l_j(t) = \prod_{\substack{k=0 \\ k \neq j}}^{2n} \left(\frac{t - t_k}{t_j - t_k}\right) \cdot \left(\frac{t_j}{t}\right)^n, j = \overline{0, 2n}, t \in \Gamma.$$

Then SNE (3) can be written as an equation

$$A_n(\varphi_n) \equiv U_n \Phi[t; \varphi_n(t); S_\tau h(t, \tau; \varphi_n(\tau))] = U_n f \quad (t \in \Gamma), \qquad (6)$$

where $A_n$ is a nonlinear operator on the surface $X_n(\Gamma)$ of polynomial functions of the form (2) with the same norm as in $H_\beta(\Gamma)$. By the same conditions 1) of the theorem, $A_n$ is also Freshet differentiable at every point

$\varphi_n^0$ of $X_n$ and its derivative has the form

$$A_n'(\varphi_n^0)g_n = U_n\{\Phi_u'[t;\varphi_n^0(t);S_\tau h(t,\tau;\varphi_n^0(\tau))]\varphi_n(t)\}+$$

$$+U_n\{\Phi_v'[t;\varphi_n^0(t);S_\tau h(t,\tau;\varphi_n^0(\tau))]S_\tau[h_u'(t,\tau;\varphi_n^0(\tau))\varphi_n(\tau)]\} \qquad (7)$$

Now let the nodes $\{t_j\}_{j=0}^{2n} \subset \Gamma$ be computed in accordance with condition 6) of the theorem. Then [2, p.49]

$$\| U_n \|_\beta \leq \mu_1 + \mu_2 \cdot ln(2n+1) \qquad (8)$$

where, here and later on, $\mu_k (k = 1, 2, ...)$ are completely definite constants not depending on n.

Hence, using (8), the linear operator $A_n'(\varphi_n^0)$ defined in (7) satisfies Lipshitz condition (5) with the new constant

$$\| A_n'(z_n^1) - A_n'(z_n^2) \|_\beta \leq (\mu_3 + \mu_4 \cdot ln(2n+1)) \| z_n^1 - z_n^2 \|_\beta \qquad (9)$$

Let $\varphi_n^*(t) (\in X_n)$ be the best uniform approximation polynomial of $\varphi(t)$. Find the estimation of $\| A'(\varphi) - A'(\varphi_n^*) \|_\beta$:

$$\| A'(\varphi)g(t) - A'(\varphi_n^*)g(t) \|_\beta \leq \| \{\Phi_u'[t;\varphi(t);S_\tau h(t,\tau;\varphi(\tau))]-$$

$$-\Phi_u'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi_n^*(\tau))]\}g(t) \|_\beta +$$

$$+ \| \Phi_v'[t;\varphi(t);S_\tau h(t,\tau;\varphi(\tau))] \cdot S_\tau[h_u'(t,\tau;\varphi(\tau))g(\tau)]-$$

$$-\Phi_v'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi_n^*(\tau))] \cdot S_\tau[h_u'(t,\tau;\varphi_n^*(\tau))g(\tau)] \|_\beta \leq$$

$$\leq \| \{\Phi_u'[t;\varphi(t);S_\tau h(t,\tau;\varphi(\tau))] - \Phi_u'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi(\tau))]\}g(t) \|_\beta +$$

$$+ \| \{\Phi_u'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi(\tau))] - \Phi_u'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi_n^*(\tau))]\}g(t) \|_\beta +$$

$$+ \| \Phi_v'[t;\varphi(t);S_\tau h(t,\tau;\varphi(\tau))] \cdot S_\tau[h_u'(t,\tau;\varphi(\tau))g(\tau)]-$$

$$-\Phi_v'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi(\tau))]S_\tau[h_u'(t,\tau;\varphi(\tau))g(\tau)] \|_\beta +$$

$$+ \| \Phi_v'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi(\tau))]S_\tau[h_u'(t,\tau;\varphi(\tau))g(\tau)]-$$

$$-\Phi_v'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi_n^*(\tau))]S_\tau[h_u'(t,\tau;\varphi(\tau))g(\tau)] \|_\beta +$$

$$+ \| \Phi_v'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi_n^*(\tau))]S_\tau[h_u'(t,\tau;\varphi(\tau))g(\tau)]-$$

$$-\Phi_v'[t;\varphi_n^*(t);S_\tau h(t,\tau;\varphi_n^*(\tau))]S_\tau[h_u'(t,\tau;\varphi_n^*(\tau))g(\tau)] \|_\beta$$

From condition 1) of the theorem, lemma 6.1 [2, p.55] and boundness of the operator $S_\tau$ on $H_\beta(\Gamma)$ we get

$$\| A'(\varphi) - A'(\varphi_n^*) \|_\beta \leq \frac{\mu_5}{n^{r+\sigma(\alpha)-\beta}} H(\varphi^{(r)};\sigma(\alpha)).$$

Hence from the reversibility of $A'(\varphi)$ and from Banach theorem it follows that for all n ($n \geq n_1$) such as

$$\frac{\mu_5}{n^{r+\sigma(\alpha)-\beta}} H(\varphi^{(r)};\sigma(\alpha)) \parallel [A'(\varphi)]^{-1} \parallel_\beta = d_n \leq q < 1,$$

the operator $A'(\varphi_n^*)$ is also reversible in $H_\beta(\Gamma)$, and

$$\parallel [A'(\varphi_n^*)]^{-1} \parallel_\beta \leq \frac{\parallel [A'(\varphi)]^{-1} \parallel_\beta}{1-d_n} \leq \frac{\parallel [A'(\varphi)]^{-1} \parallel_\beta}{1-q} = \mu_6.$$

By the theorem about the continuity of the function index [6, p.101], chainging there $\varphi$ on $\varphi_n^*$, the conditions 3)-5) of the theorem are fullfilled, as $n \geq n_1$. From (6) it follows that the operator $A_n'(\varphi_n^*)$ is the operator of collocation method for the operator $A'(\varphi_n^*)$. Thus for the operators $A'(\varphi_n^*)$ and $A_n'(\varphi_n^*)$ all the conditions of the theorem 8.1 [2, p.75] are fullfilled. Therefore for sufficiently large $n (n \geq n_2 \geq n_1)$ the operator $A_n'(\varphi_n^*)$ is reversible in $X_n$ and the estimation holds

$$\parallel [A_n'(\varphi_n^*)]^{-1} \parallel_\beta \leq \mu_7. \tag{10}$$

From the estimation (9) and (10) it follows that for sufficiently small $r_1$ the inequality

$$\sup_{\parallel \psi_n - \varphi_n^* \parallel_\beta \leq r_1} \parallel [A_n'(\varphi_n^*)]^{-1} [A_n'(\psi_n) - A_n'(\varphi_n^*)] \parallel_\beta \leq$$

$$\leq \mu_7(\mu_3 + \mu_4 \cdot \ln(2n+1)) r_1 = q < 1, \tag{11}$$

holds, where $\psi_n \in X_n$.

Let $f(t) \in H_\alpha^r(\Gamma)$. Using the well known formula of finite increments for operators, the estimation (10) and lemma 6.1 [2, p.55], we find that for all $n$ begining with a certain one ($n \geq n_1 \geq n_2$) the inequality

$$\rho_n \equiv \parallel [A_n'(\varphi_n^*)]^{-1} [A_n(\varphi_n^*) - U_n A(\varphi)] \parallel_\beta \leq$$

$$\leq \mu_7(\mu_3 + \mu_4 \cdot \ln(2n+1)) \parallel A \parallel_\beta \frac{\mu_8}{n^{r+\sigma(\alpha)-\beta}} H(\varphi^{(r)};\sigma(\alpha)) \leq r_1(1-q). \tag{12}$$

holds.

From the estimation (11), (12) and lemma [7, p.277], putting there $T = A_n$, we obtain that the equation $A_n \varphi_n = U_n f$ has in the sphere $\parallel \psi_n - \varphi_n^* \parallel_\beta \leq r_1$ the unique solution $\varphi_n(t)$ and the estimation

$$\frac{\rho_n}{1+q} \leq \parallel \varphi_n - \varphi_n^* \parallel_\beta \leq \frac{\rho_n}{1-q}$$

is proved. This means that for $n \geq n_3$ in some neighbourhood of $(2n+1)$–dimensional point $\{y_k\}_{k=-n}^{n}(y_k : \varphi_n^*(t) = \Sigma_{k=-n}^{n} y_k t^k)$ the system of non-linear equations (3) has the unique solution $\{\alpha_k\}_{k=-n}^{n}$. For approximative solution (2) the following estimation

$$\| \varphi_n^* - \varphi_n \|_\beta \leq \frac{\rho_8}{1-q} \leq \frac{\mu_9 + \mu_{10} \cdot ln(2n+1)}{n^{r+\sigma(\alpha)-\beta}} H(\varphi^{(r)}; \sigma(\alpha))$$

is obtained. From the last inequality, triangle inequality and lemma 6.1 [2, p.55] the estimation (4) follows. The theorem is proved.

## References


[1] Mushelishvili N.I. Singular integral equations. M.: Nauka, 1968. English transl.: Noordhoff, 1972.

[2] Zolotarevski V.A. Finite dimensional methods of solving singular integral equations on closed contours of integration (Russian). Chishinau: Stiintsa, 1991.

[3] Seichiuc V.N. On convergence of collocation method and method of mechanical quadratures for singular integral equations give on Lyapunov contour (Russian) // Dif. yravnenia. 1984. V.20, No.7. P.1267-1275.

[4] Smirnov V.M., Lebedev N.A. Constructive theory of complex variable functions (Russian). M.-L.: Nauka, 1964.

[5] Gabdulhaev B.G., Gorlov V.E. Solving nonlinear singular equations by reduction method (Russian) // Izv. vuzov. Matem. 1986, No.2. P.3-13.

[6] Gahov F.D. Boundary-value problems (Russian). M.: Nauka, 1977.

[7] Krasnoselski M.A., Vainikko G.M., Zabreiko P.P., Rutitski Ia.B., Stetsenko B.Ia. Approximate solution of operator equations (Russian). M.: Nauka, 1969.


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