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ON THE LYAPUNOV QUANTITIES OF TWO-DIMENSIONAL AUTONOMOUS SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A CRITICAL POINT OF CENTRE OR FOCUS TYPE

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Abstract

Analytical two-dimensional system of differential equations with a weak focus are considered. It is shown how by using a given sequence of homogeneous polynomials of even degree one can construct a function with the rate of change along trajectories of system to be a linear combination of these polynomials. The coefficients of this linear combination are the Lyapunov quantities.

1 Introduction

In this paper, we consider the system of differential equations

$$\frac{dx}{dt} = y + P(x,y), \quad \frac{dy}{dt} = -x - Q(x,y), \tag{1.1}$$

where P and Q are holomorphic functions defined in some neighbourhood of $O(0,0)$, i.e.

$$P(x,y) = \sum_{k=2}^{\infty} p_k(x,y), \quad Q(x,y) = \sum_{k=2}^{\infty} q_k(x,y) \tag{1.2}$$

(p_k, q_k are homogeneous polynomials of degree k). The origin $O(0,0)$ is either a focus or a centre (i.e. weak focus) for (1.1) [1, 2]. We face the problem of the distinction. Let us survey the classical results concerning this problem: for the infinite system of homogeneous polynomials of even degree

$$\{(x^2+y^2)^k\}_{k=1}^{\infty} \tag{1.3}$$

there exists a function

$$U(x,y) = x^2 + y^2 + \sum_{k=3}^{\infty} f_k(x,y),$$

where $f_k(x,y)$ are homogeneous polynomials of degree k, and constants $V_1, V_2, \ldots$, such that

$$\frac{dU}{dt} = \sum_{k=2}^{\infty} V_{k-1}(x^2+y^2)^k, \tag{1.4}$$

i.e. the derivative of $U(x,y)$ along trajectories of differential system (1.1) is a linear combination of polynomials (1.3). The constants $V_1, V_2, \ldots$, are polynomials in coefficients of (1.1) and are called the Lyapunov quantities. Algorithms of their computation can be found in [3–9]. If all V_k, $k=\overline{1,\infty}$, vanish then the origin $O(0,0)$ is a centre for (1.1), in the opposite case $O(0,0)$ is a focus. Let $V_k = 0, k = \overline{1, m-1}$ and $V_m \neq 0$. It follows from

(1.4) that the dU/dt is of constant sign in a neighbourhood of the origin and of the same as V_m. The origin $O(0,0)$ is asymptotically stable if $V_m < 0$ and asymptotically instable if $V_m > 0$.

It should be observed here the following properties of polynomials (1.3):

1) the even degree;

2) the integral of them taken along the unit circle is not equal to zero;

3) the constant sign.

In this paper it will be shown that condition 3) can be neglected in some cases. Let us consider arbitrary system of polynomials $\{\psi_{2k}(x,y)\}_{k=1}^{\infty}$ instead of (1.3), satisfying the properties 1) and 2). For such system (1.3) we construct a function $U(x,y)$ with the rate of change along trajectories of (1.1) to be a linear combination of polynomials $\psi_{2k}(x,y)$. The first non-zero coefficient L_m of this linear combination is a Lyapunov's quantity with the same index (up to a non-zero multiplier). This enables us to solve the problem on asymptotic stability of the critical point $O(0,0)$, even if, the total derivative

$$\frac{dU}{dt} = L_m\psi_{2m+2} + L_{m+1}\psi_{2m+4} + \ldots$$

can cease to be of the constant sign in some neighbourhood of $O(0,0)$. This happens, for example, if we consider polynomials of even degree $\psi_{2k}(x,y) = (2x^2 - y^2)(x^2+y^2)^{k-1}$, $k = \overline{1,\infty}$, such that

$$\int_0^{2\pi} \psi_{2k}(cos\varphi, sin\varphi)d\varphi = \int_0^{2\pi}(2cos^2\varphi - sin^2\varphi)d\varphi = \pi \neq 0.$$

In the last two sections of this paper a function $\mu(x,y)$ is constructed such that

$$\frac{\partial}{\partial x}\Big[\mu(y+P)\Big] - \frac{\partial}{\partial y}\Big[\mu(x+Q)\Big] = \sum_{k=1}^{\infty}\Lambda_k\psi_{2k}(x,y)$$

and the relations between Λ_k and V_k are established.

2 System (1.1) in polar coordinates

Let $x = \rho cos\varphi$, $y = \rho sin\varphi$. The system (1.1) in variables ρ, φ looks like

$$\frac{d\rho}{dt} = \sum_{k=2}^{\infty} u_k(\varphi)\rho^k, \qquad \frac{d\varphi}{dt} = -1 - \sum_{k=1}^{\infty} v_k(\varphi)\rho^k, \tag{2.1}$$

where

$$u_k(\varphi) = p_k(cos\varphi, sin\varphi)cos\varphi - q_k(cos\varphi, sin\varphi)sin\varphi,$$
$$v_k(\varphi) = p_{k+1}(cos\varphi, sin\varphi)sin\varphi + q_{k+1}(cos\varphi, sin\varphi)cos\varphi.$$

The right-hand sides of system (2.1) are holomorphic functions in some neighbourhood of $\rho = 0$ for all $\varphi \in (-\infty, +\infty)$. Replacing system (2.1) by one equation, we get

$$\frac{d\rho}{d\varphi} = -\frac{\sum_{k=2}^{\infty} u_k(\varphi)\rho^k}{1+\sum_{k=1}^{\infty} v_k(\varphi)\rho^k}. \tag{2.2}$$

The solution of equation (2.2) satisfying the initial condition $\rho = c$ for $\varphi = 0$ can be represented in the form

$$\rho(\varphi, c) = c + \sum_{k=2}^{\infty} h_k(\varphi)c^k. \tag{2.3}$$

Moreover, this series and its derivative are uniformly convergent in some domain $0 \le \varphi \le 2\pi$, $0 \le c \le c_0$ $(c_0 > 0)$. Furthermore, the functions $h_2(\varphi), h_3(\varphi), \ldots$ satisfy the conditions $h_k(0) = 0$, $k = 2, 3, 4, \ldots$.

The function $\rho(c) = \rho(2\pi, c) = c + g_2c^2 + g_3c^3 + \cdots$, where $g_k = h_k(2\pi)$, is called the return map on $0 \le c \le c_0$ of the positive semi-axis Ox. The coefficients g_k, $k = 2, 3, 4, \ldots$ of the return map are called the focal values. The index of the first nonvanishing focal value is necessarily odd [1]. If there exists such a number m that $g_{2m+1} \neq 0$ then the critical point $O(0,0)$ is a focus. In the opposite case $O(0,0)$ is a centre.

3 System (1.1) in complex variables

Let

$$Q(x,y) = \sum_{j+l \in A} c_{jl} x^j y^l, \quad P(x,y) = \sum_{j+l \in A} b_{jl} x^j y^l, \tag{3.1}$$

where c_{jl}, b_{jl} are real numbers and A is a set of different nonnegative integer numbers. Let us denote $w = x + iy$, $i^2 = -1$. Then

$$x = (\overline{w} + w)/2, \quad y = i(\overline{w} - w)/2 \tag{3.2}$$

and

$$Q + iP \equiv \sum_{j+l \in A} v_{jl} x^j y^l \equiv \sum_{j+l \in A} 2^{-j-l} z_{jl} \overline{w}^j w^l \quad (v_{jl} = c_{jl} + ib_{jl}). \tag{3.3}$$

The following relations [3, page 20, 21]

$$z_{n-j,j} = \sum_{l=0}^{n} i^l R_{jl}^{(n)} v_{n-l,l}\,, \tag{3.4}$$

$$2^n i^l v_{n-l,l} = \sum_{j=0}^{n} R_{lj}^{(n)} z_{n-j,j}\,, \tag{3.5}$$

between the coefficients z_{jl} and v_{jl} hold, where

$$R_{jl}^{(n)} = \sum_{\sigma} (-1)^\sigma \binom{l}{\sigma} \binom{n-l}{j-\sigma}. \tag{3.6}$$

The summation is over all integers σ, which satisfy the conditions $0 \le \sigma \le l$, $0 \le j - \sigma \le n - l$ and $\binom{l}{\sigma}$ denotes the binomial coefficients, i.e. $\binom{l}{\sigma} = \frac{l!}{\sigma!(l-\sigma)!}$. The number $R_{jl}^{(n)}$ represents the coefficient by t^j in expression $(1-t)^l(1+t)^{n-l}$. It is known that

$$R_{n-j,l}^{(n)} = (-1)^l R_{jl}^{(n)}. \tag{3.7}$$

Let $z_{n-j,j} = C_j^{(n)} + iB_j^{(n)}$; $a_{jl} = b_{jl}$, $a_{jl}^* = c_{jl}$ if l is even and $a_{jl} = c_{jl}$, $a_{jl}^* = b_{jl}$ if l is odd. It follows from (3.5) and (3.4) that

$$2^{n+1} b_{n-l,l} = i^{l+1} \sum_{j=0}^{n} R_{lj}^{(n)} \left[\overline{z}_{n-j,j} - (-1)^l z_{n-j,j} \right], \tag{3.8}$$

$$2^{n+1} c_{n-l,l} = i^{l} \sum_{j=0}^{n} R_{lj}^{(n)} \left[\overline{z}_{n-j,j} + (-1)^l z_{n-j,j} \right]. \tag{3.9}$$

$$G_j^{(n)} = \sum_{l=0}^{n} r_{jl}^{(n)} a_{n-l,l}^{*}, \quad B_j^{(n)} = \sum_{l=0}^{n} (-1)^l r_{jl}^{(n)} a_{n-l,l},$$

where $r_{jl}^{(n)} = (-1)^{[\frac{l+1}{2}]} R_{jl}^{(n)}$ (see [3, page 22, 23]).

Let us take $P \equiv 0$ in (3.3). Then $Q(x,y) = \sum_{j+l \in A} 2^{-j-l} z_{jl} \overline{w}^j w^l$,

$$\overline{z}_{jl} = z_{lj}. \tag{3.10}$$

Indeed, according to (3.4) and (3.7), we have

$$\begin{aligned} \overline{z}_{n-j,j} &= \sum_{l=0}^{n} \overline{i^l R_{jl}^{(n)} v_{n-l,l}} = \sum_{l=0}^{n} \overline{i}^l R_{jl}^{(n)} c_{n-l,l} \\ &= \sum_{l=0}^{n} (-1)^l i^l R_{jl}^{(n)} c_{n-l,l} = \sum_{l=0}^{n} i^l R_{n-j,l}^{(n)} c_{n-l,l} = z_{j,n-j}. \end{aligned}$$

Hence, by carring out the change (3.2) in any series (polynomial) $Q(x,y)$ with real coefficients, we obtain the series $Q^*(\overline{w},w)$. For coefficients of $Q^*(\overline{w},w)$ the equality (3.10) are fulfilled. It follows from (3.10) that the coefficient z_{ll} by $\overline{w}^l w^l$ is a real number. The converse statement is also known to be true, i.e., if the coefficients of the series (polynomial) $Q^*(\overline{w},w)$ satisfy the equality (3.10), then replacing in $Q^*(\overline{w},w)$ $\overline{w}$ by $x-iy$ and w by $x+iy, i^2=-1$, we obtain a series in x and y with real coefficients.

Let $Q(x,y)$ be a homogeneous polynomial of degree $2k$ with real coefficients and

$$Q_{2k}^*(\overline{w},w) = Q_{2k}\left(\frac{\overline{w}+w}{2}, \frac{i(\overline{w}-w)}{2}\right) = 2^{-2k} \sum_{j+l=2k} z_{jl} \overline{w}^j w^l.$$

Then

$$\int_0^{2\pi} Q_{2k}(cos\varphi, sin\varphi) d\varphi = 2^{-2k+1} \pi z_{kk}.$$

Let us consider the following three particular case:

1) $Q_{2k}(x,y) = (x^2+y^2)^k$. In this case $Q_{2k}^*(\overline{w},w) = \overline{w}^k w_k$. Therefore $z_{kk} = 2^{2k}$ and

$$\int_0^{2\pi} (cos^2\varphi + sin^2\varphi)^k d\varphi = 2^{-2k+1} \pi 2^{2k} = 2\pi.$$

2) $Q_{2k}(x,y) = x^{2k}$. Assuming in (3.4) that $n = 2k$, $v_{2k,0} = 1$, $v_{2k-l,l} = 0$ for $l \neq 0$, we get $z_{kk} = R_{k,0}^{(2k)}$. From (3.6) $R_{k,0}^{(2k)} = \binom{2k}{k} = \frac{(2k)!}{k!k!}$. Consequently

$$\begin{aligned} \int_0^{2\pi} cos^{2k}\varphi d\varphi &= 2^{-2k+1}\pi \frac{(2k)!}{k!k!} = \frac{1\cdot 2\cdot 3\cdots(2k-1)\cdot 2k}{2^{2k}k!k!} 2\pi \\ &= \frac{2^k k!(2k-1)!!}{2^k k! 2^k k!} 2\pi = \frac{(2k-1)!!}{(2k)!!} 2\pi. \end{aligned}$$

3) $Q_{2k}(x,y) = y^{2k}$. From (3.4) and (3.6) we have, that

$$z_{kk} = (-1)^k R_{k,2k}^{(2k)} = (-1)^k (-1)^k \binom{2k}{k} = \frac{(2k)!}{k!k!}.$$

Then

$$\int_0^{2\pi} sin^{2k}\varphi d\varphi = \frac{(2k-1)!!}{(2k)!!} 2\pi.$$

Let $A = \{2, 3, 4, 5, \dots\}$. Taking into account (3.1), the system (1.1) can be written in the form

$$\frac{dx}{dt} = y + \sum_{j+l=2}^{\infty} b_{jl} x^j y^l, \quad \frac{dy}{dt} = -x - \sum_{j+l=2}^{\infty} c_{jl} x^j y^l. \tag{3.11}$$

In complex variables $w = x + iy$, $i^2 = -1$, system (3.11) becomes (see (3.3))

$$i\frac{dw}{dt} = w + \sum_{r=2}^{\infty} \psi_r \quad \left(i\frac{d\overline{w}}{dt} = -\overline{w} - \sum_{r=2}^{\infty} \overline{\psi_r} \right), \tag{3.12}$$

where

$$\psi_r = \sum_{j+l=r} 2^{-j-l} z_{jl} \overline{w}^j w^l. \tag{3.13}$$

The coefficients b_{jl}, c_{jl} and z_{jl} satisfy the relations (3.4), (3.5), (3.8) and (3.9).

If $A = \{2, 3\}$ then (3.11) looks as

$$\begin{aligned} \frac{dx}{dt} &= y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3, \\ \frac{dy}{dt} &= -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3), \end{aligned} \tag{3.14}$$

and the corresponding equation (3.12) is

$$i\frac{dw}{dt} = w + \frac{1}{4}(z_{20}\overline{w}^2 + z_{11}\overline{w}w + z_{02}w^2) + \frac{1}{8}(z_{30}\overline{w}^3 + z_{21}\overline{w}^2 w + z_{12}\overline{w}w^2 + z_{03}w^3), \tag{3.15}$$

where

$$\begin{aligned} z_{20} &= c_{20} - b_{11} - c_{02} + i(b_{20} + c_{11} - b_{02}), \\ z_{11} &= 2(c_{20} + c_{02}) + 2i(b_{20} + b_{02}), \\ z_{02} &= c_{20} + b_{11} - c_{02} + i(b_{20} - c_{11} - b_{02}), \\ z_{30} &= c_{30} - b_{21} - c_{12} + b_{03} + i(b_{30} + c_{21} - b_{12} - c_{03}), \\ z_{21} &= 3c_{30} - b_{21} + c_{12} - 3b_{03} + i(3b_{30} + c_{21} + b_{12} + 3c_{03}), \\ z_{12} &= 3c_{30} + b_{21} + c_{12} + 3b_{03} + i(3b_{30} - c_{21} + b_{12} - 3c_{03}), \\ z_{03} &= c_{30} + b_{21} - c_{12} - b_{03} + i(b_{30} - c_{21} - b_{12} + c_{03}). \end{aligned}$$

It should be pointed out that, in general, the problem of determining when a differential system (3.14) has a centre at a singular point is open. The problem of centre is completely solved only in the cases when the right-hand sides of system (3.14) contain either only quadratic or only cubic terms (see [3]). If the right-hand sides of (3.14) contain both quadratic and cubic terms then the abovementioned problem is solved only in some particular cases (see, for example, [9–21]).

4 Construction of the function $U(x, y)$

Let $\psi_{2k}(x, y)$, $k = 2, 3, 4, \dots$, be real homogeneous polynomials of degree $2k$, such that

$$\int_0^{2\pi} \psi_{2k}(\cos\varphi, \sin\varphi)\, d\varphi \neq 0 \quad \forall k. \tag{4.1}$$

Theorem 4.1 *There exists a formal power series*

$$U(x,y) = x^2 + y^2 + f_3(x,y) + f_4(x,y) + \cdots \tag{4.2}$$

and constants L_m, $m = \overline{1,\infty}$, such that the rate of change of U along trajectories of (1.1) ($\equiv$ (3.11)) is

$$\frac{dU}{dt} = \frac{\partial U}{\partial x}(y+P) - \frac{\partial U}{\partial y}(x+Q) \equiv \sum_{k=2}^{\infty} L_{k-1}\psi_{2k}(x,y). \tag{4.3}$$

To prove this statement we pass to complex variables ($w = x + iy$, $i^2 = -1$). Let us put

$$F_{k+1}(\overline{w},w) = f_{k+1}\left(\frac{\overline{w}+w}{2}, \frac{i(\overline{w}-w)}{2}\right), \quad \Psi_{2k}(\overline{w},w) = \psi_{2k}\left(\frac{\overline{w}+w}{2}, \frac{i(\overline{w}-w)}{2}\right),$$

$$F(\overline{w},w) = U\left(\frac{\overline{w}+w}{2}, \frac{i(\overline{w}-w)}{2}\right) = \sum_{k=2}^{\infty} F_k(\overline{w},w), \quad F_2(\overline{w},w) = \overline{w}w, \quad k = \overline{2,\infty}.$$

If

$$F_k(\overline{w},w) = \sum_{j+l=k} a_{jl}\overline{w}^j w^l, \; k = 2,3,\ldots, \tag{4.4}$$

($a_{20} = a_{11} - 1 = a_{02} = 0$), then $F(\overline{w},w) = \overline{w}w + \sum\limits_{j+l=3}^{\infty} a_{jl}\overline{w}^j w^l$ and $\overline{a}_{jl} = a_{lj}$. We shall assume that

$$\Psi_{2k}(\overline{w},w) = \sum_{j+l=2k} \psi_{jl}\overline{w}^j w^l, \quad \overline{\psi_{jl}} = \psi_{lj}. \tag{4.5}$$

In new variables the system (4.3) can be written in the form (see(3.12)):

$$\frac{dU}{dt} = \frac{dF}{dt} = F'_{\overline{w}}\frac{d\overline{w}}{dt} + F'_{w}\frac{dw}{dt} \equiv \left(\sum_{k=2}^{\infty} F'_{k\overline{w}}\right)\left(i\overline{w} + i\sum_{r=2}^{\infty}\overline{\varphi}_r\right) -$$

$$\left(\sum_{k=2}^{\infty} F'_{kw}\right)\left(iw + i\sum_{r=2}^{\infty}\varphi_r\right) \equiv \sum_{k=2}^{\infty} L_{k-1}\Psi_{2k}(\overline{w},w).$$

This implies that

$$\sum_{k=2}^{\infty}\left[F'_{k\overline{w}}\left(\overline{w} + \sum_{r=2}^{\infty}\overline{\varphi}_r\right) - F'_{kw}\left(w + \sum_{r=2}^{\infty}\varphi_r\right)\right] \equiv -\sum_{k=2}^{\infty} L_{k-1}\Psi_{2k}(\overline{w},w).$$

Identifying the terms of degree k in $\overline{w}$ and w in the last identity, we get

$$\overline{w}F'_{k\overline{w}} - wF'_{kw} \equiv \sum_{r=2}^{k-1}\left(F'_{rw}\varphi_{k-r+1} - F'_{r\overline{w}}\overline{\varphi}_{k-r+1}\right)$$

if k is odd ($k = 3,5,7,\ldots$) and

$$\overline{w}F'_{k\overline{w}} - wF'_{kw} \equiv \sum_{r=2}^{k-1}\left(F'_{rw}\varphi_{k-r+1} - F'_{r\overline{w}}\overline{\varphi}_{k-r+1}\right) - iL_{\frac{k}{2}-1}\Psi_k(\overline{w},w)$$

if k is even ($k = 4,6,8,\ldots$). Now identifying the coefficients of $\overline{w}$ and w, we obtain (see (4.4), (3.13), (4.5)) that

$$(j-l)a_{jl} = 2^{-j-l}\sum_{p+q=2}^{j+l-1} 2^{p+q-1} q(a_{pq}z_{j-p,l-q+1} - \overline{a}_{pq}\overline{z}_{l-p,j-q+1}) \tag{4.6}$$

if $j+l$ is odd and

$$(j-l)z_{jl} = 2^{-j-l}\sum_{p+q=2}^{j+l-1} 2^{p+q-1} q(a_{pq} z_{j-p,l-q+1} - \bar{a}_{pq}\bar{z}_{l-p,j-q+1}) - iL_{\frac{j+l-2}{2}}\psi_{jl} \qquad (4.7)$$

if $j+l$ is even. If $j=l$, then it follows from (4.7) that

$$2^{-2l+1} i\,Im \sum_{p+q=2}^{2l-1} 2^{p+q-1} q a_{pq} z_{l-p,l-q+1} - iL_{l-1}\psi_{ll} = 0.$$

This implies that

$$L_{l-1} = 2^{-2l+1}\psi_{ll}^{-1}\,Im \sum_{\substack{p+q=2\\(p\le l,\ q\le l+1)}}^{2l-1} 2^{p+q-1} q a_{pq} z_{l-p,l-q+1} \quad (l=2,3,4,\dots). \qquad (4.8)$$

Assume that the coefficients a_{ll}, $l=2,3,4,\dots$, take on arbitrary real values. The formulas (4.6), (4.8) and (4.7) allow us to determine step by step coefficients of the function $F(\overline{w},w)$ and hence the coefficients of the function (4.2). Thus, taking into account that $a_{20}=a_{11}-1=a_{02}=0$, from (4.6) for $j+l=3$, we find that

$$a_{30} = \frac{1}{12}z_{20}, \quad a_{21} = \frac{1}{4}(z_{11}-\bar{z}_{02}), \quad a_{12} = \frac{1}{4}(\bar{z}_{11}-z_{02}), \quad a_{03} = \frac{1}{12}\bar{z}_{20},$$

and from (4.8) when $l=2$ we find

$$L_1 = 2^{-3}\psi_{22}^{-1}\,Im(2z_{12} - z_{11}z_{02}). \qquad (4.9)$$

Hence, the function $F(\overline{w},w)$ is formally found up to the terms of the form $a_{ll}\overline{w}^l w^l$ ($l=2,3,4,\dots$). Usually, it is supposed that $a_{ll}=0$ ($l=2,3,4,\dots$).

The values $L_1, L_2, \dots, L_n, \dots$ will be called the Lyapunov's quantities or $L-$values corresponding to the system of homogeneous polynomials $\{\psi_{2k}(x,y),\ k=2,3,4,\dots\}$.

Note that equality (4.8) for $\psi_{2k}(x,y)=(x^2+y^2)^k$, $k=\overline{2,\infty}$, becomes

$$V_{l-1} = 2^{-2l+1}\,Im \sum_{\substack{p+q=2\\(p\le l,\ q\le l+1)}}^{2l-1} 2^{p+q-1} q a_{pq} z_{l-p,l-q+1} \quad (l=2,3,4,\dots). \qquad (4.10)$$

5 Relation between $L-$values and focal values

Let us take function (4.2) satisfying (4.3) with $L_k=0$ for $k<l-2$ and $L_{l-1}\neq 0$, and construct the polynomial (see [21, page 14-15])

$$U_1(x,y) = x^2+y^2+\sum_{k=3}^{2l} f_k(x,y).$$

The derivative of the polynomial $U_1(x,y)$ along orbits of system (1.1) ($\equiv$(3.11)) is

$$\frac{dU_1}{dt} = L_{l-1}\psi_{2l}(x,y) + \varphi_1(x,y),$$

where $\varphi_1(x,y)$ is a holomorphic function in some neighbourhood of $O(0,0)$ and $\varphi_1(x,y) = o((x^2+y^2)^l)$ as $x^2+y^2\to 0$. Using polar coordinates ρ,φ, we obtain

$$U_1(\rho cos\varphi, \rho sin\varphi) = \rho^2 + \sum_{k=3}^{2l} f_k(\rho cos\varphi, \rho sin\varphi)$$

and by (2.1)

$$\frac{dU_1}{d\varphi} = \frac{dU_1}{dt}\frac{dt}{d\varphi} = \left[\rho^{2l}L_{l-1}\psi_{2l}(cos\varphi, sin\varphi) + \varphi_1(\rho cos\varphi, sin\varphi)\right]\times \\ \times\left[-1 - \sum_{k=1}^{\infty} v_k(\varphi)\rho^k\right]^{-1}. \tag{5.1}$$

Let (2.3) be solution of (2.2) with initial condition $\rho(0,c) = c$. By substituting this solution in (5.1) and integrating (5.1) with respect to φ 0 to 2π, we find that

$$\rho^2(2\pi,c) - \rho^2(0,c) + \sum_{k=3}^{2l}\left\{f_k[\rho(2\pi,c),0] - f_k[\rho(0,c),0]\right\} \equiv \\ \equiv \int_0^{2\pi}\left\{-L_{l-1}\psi_{2l}(cos\varphi, sin\varphi)\left[c + \sum_{k=2}^{\infty} h_k(\varphi)c^k\right]^{2l} + \cdots\right\}d\varphi. \tag{5.2}$$

Since the right-hand side of the identity (5.2) (with respect to c) is $-L_{l-1}\psi_0 c^{2l} + \cdots$ and is different from zero for small c, then as it is seen from the expression of left-hand side of (5.2) $\rho(2\pi,c) \neq \rho(0,c) = c$. So, there exists such m, that $h_2(2\pi) = h_3(2\pi) = \cdots = h_{m-1}(2\pi) = 0$ and $h_m(2\pi) \neq 0$. Let us write the identity (5.2) as

$$(c + h_m(2\pi)c^m + \cdots)^2 - c^2 + \sum_{k=3}^{2l}\left\{f_k[c + h_m(2\pi)c^m + \cdots, 0] - f_k(c,0)\right\} \equiv \\ \equiv -2\pi L_{l-1}\psi_0 c^{2l} + \cdots$$

or

$$2h_m(2\pi)c^{m+1} + \cdots \equiv -2\pi L_{l-1}\psi_0 c^{2l} + \cdots.$$

This implies that $m = 2l-1$ and $h_{2l-1}(2\pi) = -\pi L_{l-1}\psi_0$. Now, taking into account that $g_{2l-1} = h_{2l-1}(2\pi)$ is the $(2l-1)$-th focal value and using the above results, it is easy to see that the first nonvanishing focal value has an odd index.

Thus, if the first $l-2$ $(l \geq 2)$ of L-values vanish and the $(l-1)$-th one is different from zero, then the first $2l-2$ focal values also vanish and the $(2l-1)$-th one is different from zero. The converse statement follows from the same identity (5.2). Moreover, between L_{l-1} and g_{2l-1} there exists the following relation

$$g_{2l-1} = -\pi L_{l-1}\psi_0. \tag{5.3}$$

In the case, when $\psi_{2k}(x,y) = (x^2+y^2)^k$ $(\psi_{2k}(x,y) = x^{2k}$ or $\psi_{2k}(x,y) = y^{2k})$, $k = 2,3,4,\ldots$, we get

$$g_{2l-1} = -\pi L_{l-1} \quad \left(g_{2l-1} = -\frac{(2l-1)!!}{(2l)!!}\pi L_{l-1}\right).$$

Thus, we arrive at the following conclusion: the existence of a centre at the origin for (1.1) is equivalent to the vanishing of all L-values. From (4.8) we obtain the following centre conditions

$$Im\sum_{p+q=2}^{2l-1} 2^{p+q-1} q a_{pq} z_{l-p,l-q+1} = 0 \quad (l = 2,3,\ldots). \tag{5.4}$$

For the cubic system (3.14) (equation (3.15)) the conditions (5.4) look as:

$$Im\Big[2(l+1)a_{l-2,l+1}z_{20} + 2la_{l-1,l}z_{11} + 2(l-1)a_{l,l-1}z_{02} + (l+1)a_{l-3,l+1}z_{30} + \\ la_{l-2,l}z_{21} + (l-1)a_{l-1,l-1}z_{12} + (l-2)a_{l,l-2}z_{03}\Big] = 0 \quad (l = 2,3,\ldots),$$

where $a_{00} = a_{10} = a_{01} = a_{20} = a_{11} - 1 = a_{02} = 0$; $a_{\alpha\beta} = 0$ by $\alpha < 0$ or $\beta < 0$; $a_{\beta\beta} = 0$ by $\beta > 1$ and (see (4.6), (4.7))

$$
\begin{aligned}
a_{\alpha\beta} = & \frac{1}{8(\alpha-\beta)}\Big[2(\beta+1)a_{\alpha-2,\beta+1}z_{20} + 2\beta a_{\alpha-1,\beta}z_{11} + 2(\beta-1)a_{\alpha,\beta-1}z_{02} \\
+ & (\beta+1)a_{\alpha-3,\beta+1}z_{30} + \beta a_{\alpha-2,\beta}z_{21} + (\beta-1)a_{\alpha-1,\beta-1}z_{12} \\
+ & (\beta-2)a_{\alpha,\beta-2}z_{03} - 2(\alpha+1)a_{\alpha+1,\beta-2}\bar{z}_{20} - 2\alpha a_{\alpha,\beta-1}\bar{z}_{11} \\
- & 2(\alpha-1)a_{\alpha-1,\beta}\bar{z}_{02} - (\alpha+1)a_{\alpha+1,\beta-3}\bar{z}_{30} - \alpha a_{\alpha,\beta-2}\bar{z}_{21} \\
- & (\alpha-1)a_{\alpha-1,\beta-1}\bar{z}_{12} - (\alpha-2)a_{\alpha-2,\beta}\bar{z}_{03}\Big]
\end{aligned}
$$

by $\alpha + \beta = 3, 4, \ldots,\ \alpha \geq 0,\ \beta \geq 0,\ \alpha \neq \beta$.

In the case, when four coefficients of equation (3.15) are equal to zero, the problem of a centre is solved [20]. Let us tabulate this result.

No	Zero coefficients of the equation (3.15)	Necessary and sufficient conditions of existence a centre
1	2	3
1	$z_{30} = z_{21} = z_{12} = z_{03} = 0$	a) $Im(z_{11}z_{02}) = Im(z_{20}\bar{z}_{11}^{3}) = 0$ or b) $z_{11}(z_{11} - 2\bar{z}_{02}) = 0$ or c) $z_{02} + 2\bar{z}_{11} = z_{20}\bar{z}_{20} - z_{11}\bar{z}_{11} = 0$.
2	$z_{02} = z_{21} = z_{12} = z_{03} = 0$	$Im(z_{30}\bar{z}_{11}^{3}) = z_{11}Im(z_{20}^{4}\bar{z}_{30}^{3}) = Im(z_{11}^{4}\bar{z}_{30}) = 0$
3	$z_{02} = z_{30} = z_{12} = z_{03} = 0$	$Im(z_{20}\bar{z}_{11}^{3}) = Im(z_{20}^{2}\bar{z}_{21}^{3}) = z_{20}Im(z_{11}^{2}\bar{z}_{21}) = 0$
4	$z_{02} = z_{30} = z_{21} = z_{03} = 0$	$Im(z_{20}\bar{z}_{11}^{3}) = Im(z_{12}) = 0$
5	$z_{02} = z_{30} = z_{21} = z_{12} = 0$	$Im(z_{20}\bar{z}_{11}^{3}) = Im(z_{20}^{2}z_{03}^{2}) = Im(z_{11}^{2}z_{03}) = 0$
6	$z_{11} = z_{21} = z_{12} = z_{03} = 0$	$z_{30}Im(z_{20}z_{02}^{2}) = z_{02}Im(z_{20}^{4}\bar{z}_{30}^{3}) = Im(z_{02}^{4}z_{30}) = 0$
7	$z_{11} = z_{30} = z_{12} = z_{03} = 0$	$z_{21}Im(z_{20}z_{02}^{2}) = Im(z_{20}^{2}\bar{z}_{21}^{3}) = Im(z_{02}^{2}z_{21}) = 0$
8	$z_{11} = z_{30} = z_{21} = z_{03} = 0$	a) $z_{12} = 0$ or b) $Im(z_{20}z_{02}^{2}) = Im(z_{12}) = 0$
9	$z_{11} = z_{30} = z_{21} = z_{12} = 0$	$z_{03}Im(z_{20}z_{02}^{2}) = Im(z_{20}^{2}z_{03}^{3}) = z_{20}Im(z_{02}^{2}\bar{z}_{03}) = 0$
10	$z_{20} = z_{21} = z_{12} = z_{03} = 0$	a) $z_{11} - 2\bar{z}_{02} = 0$ or b) $Im(z_{11}z_{02}) = Im(z_{11}^{3}\bar{z}_{30}) = Im(z_{02}^{3}z_{30}) = 0$
11	$z_{20} = z_{30} = z_{12} = z_{03} = 0$	$Im(z_{11}z_{02}) = z_{02}Im(z_{11}^{2}\bar{z}_{21}) = Im(z_{02}^{2}z_{21}) = 0$
12	$z_{20} = z_{30} = z_{21} = z_{03} = 0$	a) $z_{11}z_{02} - 2z_{12} = 0$ or b) $Im(z_{11}z_{02}) = Im(z_{12}) = 0$
13	$z_{20} = z_{30} = z_{21} = z_{12} = 0$	a) $2z_{11} - \bar{z}_{02} = 0$ or b) $Im(z_{11}z_{02}) = Im(z_{11}^{3}z_{03}) = z_{11}Im(z_{02}^{2}\bar{z}_{03}) = 0$
14	$z_{11} = z_{02} = z_{12} = z_{03} = 0$	$z_{21}Im(z_{20}^{3}\bar{z}_{30}^{2}) = Im(z_{20}^{2}\bar{z}_{21}^{3}) = Im(z_{30}\bar{z}_{21}^{3}) = 0$
15	$z_{11} = z_{02} = z_{21} = z_{03} = 0$	$Im(z_{12}) = 0$

1	2	3
16	$z_{11} = z_{02} = z_{21} = z_{12} = 0$	$z_{03} Im(z_{20}^3 \bar{z}_{30}^2) = Im(z_{20}^2 \bar{z}_{03}^2)$ $= Im(z_{30} \bar{z}_{03}^2) = 0$
17	$z_{11} = z_{02} = z_{30} = z_{03} = 0$	$Im(z_{20}^3 \bar{z}_{21}^2) = Im(z_{12}) = 0$
18	$z_{11} = z_{02} = z_{30} = z_{12} = 0$	a) $(\bar{z}_{21} - 3z_{03})(\bar{z}_{21} + z_{03}) = 0$ or b) $Im(z_{20}^2 \bar{z}_{21}^3) = Im(z_{20}^2 z_{03}^3)$ $= Im(z_{21} z_{03}) = 0$
19	$z_{11} = z_{02} = z_{30} = z_{21} = 0$	$Im(z_{20}^2 z_{03}^2) = Im(z_{12}) = 0$
20	$z_{20} = z_{02} = z_{12} = z_{03} = 0$	$Im(z_{11}^4 \bar{z}_{30}) = z_{30} Im(z_{11}^2 \bar{z}_{21})$ $= Im(z_{30} \bar{z}_{21}^2) = 0$
21	$z_{20} = z_{02} = z_{21} = z_{03} = 0$	$Im(z_{11}^4 \bar{z}_{30}) = Im(z_{12}) = 0$
22	$z_{20} = z_{02} = z_{21} = z_{12} = 0$	$Im(z_{11}^4 \bar{z}_{30}) = Im(z_{11}^2 z_{03})$ $= Im(z_{30} z_{03}^2) = 0$
23	$z_{20} = z_{02} = z_{30} = z_{03} = 0$	$Im(z_{12}) = 0$
24	$z_{20} = z_{02} = z_{30} = z_{12} = 0$	$z_{03} Im(z_{11}^2 \bar{z}_{21}) = Im(z_{11}^2 z_{03})$ $= Im(z_{21} z_{03}) = 0$
25	$z_{20} = z_{02} = z_{30} = z_{21} = 0$	$Im(z_{11}^2 z_{03}) = Im(z_{12}) = 0$
26	$z_{20} = z_{11} = z_{12} = z_{03} = 0$	$Im(z_{02}^4 z_{30}) = Im(z_{02}^4 z_{21})$ $= Im(z_{30} \overline{z_{21}}^2) = 0$
27	$z_{20} = z_{11} = z_{21} = z_{03} = 0$	$Im(z_{02}^4 z_{30}) = Im(z_{12}) = 0$
28	$z_{20} = z_{11} = z_{21} = z_{12} = 0$	$Im(z_{02}^4 z_{30}) = z_{30} Im(z_{02}^2 \bar{z}_{03})$ $= Im(z_{30} z_{03}^2) = 0$
29	$z_{20} = z_{11} = z_{30} = z_{03} = 0$	$Im(z_{02}^2 z_{21}) = Im(z_{12}) = 0$
30	$z_{20} = z_{11} = z_{30} = z_{12} = 0$	a) $\bar{z}_{21} + z_{03} = 0$ or b) $Im(z_{02}^2 z_{21}) = z_{21} Im(z_{02}^2 \bar{z}_{03})$ $= Im(z_{21} z_{03}) = 0$
31	$z_{20} = z_{11} = z_{30} = z_{21} = 0$	a) $z_{12} = 0$ or b) $Im(z_{02}^3 \bar{z}_{03}) = Im(z_{12}) = 0$
32	$z_{20} = z_{11} = z_{02} = z_{03} = 0$	$Im(z_{30} \bar{z}_{21}^2) = Im(z_{12}) = 0$
33	$z_{20} = z_{11} = z_{02} = z_{12} = 0$	a) $Im(z_{21} z_{03}) = Im(z_{30} \bar{z}_{21}^2) = 0$ or b) $z_{21} - 3\bar{z}_{03} = 0$ or c) $z_{03} + 3\bar{z}_{21} = z_{30}\bar{z}_{30} - z_{21}\bar{z}_{21} = 0.$
34	$z_{20} = z_{11} = z_{02} = z_{21} = 0$	$Im(z_{30}^2 z_{03}^2) = Im(z_{12}) = 0$
35	$z_{20} = z_{11} = z_{02} = z_{30} = 0$	$Im(z_{21} z_{03}) = Im(z_{12}) = 0$

At the end of this section we shall state three remarks.

Remark 5.1 *If we normalize the system $\{\psi_{2k}(x,y)\}$ so that*

$$\frac{1}{2\pi}\int_0^{2\pi} \psi_{2k}(\cos\varphi, \sin\varphi)d\varphi = 1 \quad \forall k,$$

then $L_1 = L_2 = \cdots = L_{m-1} = 0$, $L_m \neq 0$, implies that $V_1 = V_2 = \cdots = V_{m-1} = 0$ and $V_m = L_m$.

Remark 5.2 *Let $U(x,y)$ be a formal series that*

$$\frac{dU}{dt} = \psi_{2l}(x,y) + \theta(x,y), \tag{5.5}$$

where $\theta(x,y) = 0((x^2+y^2)^l)$ as $x^2+y^2 \to 0$ and $\psi_{2l}(x,y)$ be a homogeneous polynomial of degree $2l$, $(l \geq 2)$, such that

$$\int_0^{2\pi} \psi_{2l}(\cos\varphi, \sin\varphi)d\varphi \neq 0 \tag{5.6}$$

then the origin $O(0,0)$ is a focus for (3.11) (there exists such m, $1 \leq m \leq l-1$ that $V_1 = V_2 = \cdots = V_{m-1} = 0$, $V_m \neq 0$). The asymptotic stability of the origin is determined by the sign of V_m.

Remark 5.3 *If*

$$\int_0^{2\pi} \psi_{2l}(\cos\varphi, \sin\varphi)d\varphi = 0 \tag{5.7}$$

and (5.5) are satisfied, we cannot in general conclude that $O(0,0)$ is a focus. This is confirmed by the following two examples.

Example 5.1. The following system

$$\dot{x} = y(1+x^2+y^2), \quad \dot{y} = x(-1+x^2+y^2) \tag{5.8}$$

is symmetric with respect to the axes Ox and Oy. Hence, we have a centre at the origin. The derivative of the function $U(x,y) = x^2+y^2-x^4-x^3y-xy^3+y^4$ along trajectories of (5.8) is

$$\frac{dU}{dt} = x^4 - y^4 - x^6 - 4x^5y - 7x^4y^2 - 7x^2y^4 + 4xy^5 - y^6.$$

For $\psi_4(x,y) = x^4 - y^4$ (5.7) is satisfied.

Example 5.2. The following system of differential equations

$$\dot{x} = y(1-9x^2), \quad \dot{y} = (1+y)(-x+2x^2+2xy-y^2) \tag{5.9}$$

has a focus at the origin because $g_7 \neq 0$ $(g_3 = g_5 = 0)$. The derivative of the function $U(x,y) = (3x^2+3y^2+6xy^2+2y^3+6x^4-3x^3y-3x^2y^2+9xy^3)/3$ along trajectories of system (5.9) is

$$\begin{aligned}\frac{dU}{dt} &= x^4 - y^4 - 2x^6 - 5x^4y + 25x^3y^2 + 5x^2y^3 - 9xy^4 - 2y^5 - 74x^5y \\ &+ 21x^4y^2 + 33x^3y^3 - 7x^2y^4 - 9xy^5.\end{aligned}$$

6 Construction of the function $\mu(x,y)$

Put $\tilde{P}(x,y) = y + P(x,y)$, $\tilde{Q}(x,y) = x + Q(x,y)$ (see (1.1)).

Dulac's criterion, about nonexistence of closed orbits in some neighbourhood of the origin, for system (1.1) can be formulated: If there exists a continuously differentiable function $\mu(x,y)$ in a simply connected neighbourhood G of the origin such that

$$\frac{\partial}{\partial x}\Big[\mu(x,y)\tilde{P}(x,y)\Big] - \frac{\partial}{\partial y}\Big[\mu(x,y)\tilde{Q}(x,y)\Big]$$

has constant sign and is not identically zero in any subregion, then system (1.1) has no closed orbits in G and this means that $O(0,0)$ is a focus for (1.1). In this and next sections it will be shown that Dulac's criterion can be modificated. The require that $div(\mu\tilde{P},\mu\tilde{Q})$ has constant sign can be neglected in some cases.

Let $\psi_{2k}(x,y)$, $k = 1,2,3,\ldots$, be homogeneous polynomials with real coefficients satisfying condition (4.1).

Theorem 6.1 *There exist a function*

$$\mu(x,y) = 1 + \sum_{k=1}^{\infty}\mu_k(x,y), \tag{6.1}$$

where $\mu_k(x,y)$ are homogeneous polynomials of degree k, and such constants Λ_k, $k = 1,2,3,\ldots$, that

$$\frac{\partial(\mu\tilde{P})}{\partial x} - \frac{\partial(\mu\tilde{Q})}{\partial y} \equiv \sum_{k=1}^{\infty}\Lambda_k\psi_{2k}(x,y). \tag{6.2}$$

Proof. Let us pass to complex variables $w = x + iy$, $i^2 = -1$. Denote

$$\Psi_{2k}(\overline{w},w) = \psi_{2k}\left(\tfrac{\overline{w}+w}{2}, \tfrac{i(\overline{w}-w)}{2}\right) = \sum_{j+l=2k}\psi_{jl}(\overline{w}^j w^l), \quad \overline{\psi}_{jl} = \psi_{lj},$$

$$M_k(\overline{w},w) = \mu_k\left(\tfrac{\overline{w}+w}{2}, \tfrac{i(\overline{w}-w)}{2}\right) = \sum_{j+l=k} m_{jl}\overline{w}^j w^l, \quad \overline{m}_{jl} = m_{lj},$$

$$M(\overline{w},w) = 1 + \sum_{k=1}^{\infty}M_k(\overline{w}^j w^l).$$

The identity (6.2) becomes (see (3.12))

$$\sum_{k=1}^{\infty}\Big[M'_{k\overline{w}}(\overline{w} + \sum_{r=2}^{\infty}\overline{\varphi}_r) - M'_{kw}(w + \sum_{r=2}^{\infty}\varphi_r)\Big] +$$
$$(1 + \sum_{k=1}^{\infty}M_k)\sum_{r=2}^{\infty}(\overline{\varphi}'_{r\overline{w}} - \varphi'_{rw}) \equiv -i\sum_{k=1}^{\infty}\Lambda_k\Psi_{2k}(\overline{w},w).$$

Identifying the terms of degree k in $\overline{w}$ and w, we obtain

$$\begin{aligned}(j-l)m_{jl} &= 2^{-j-l-1}\Big[(l+1)z_{j,l+1} - (j+1)\overline{z}_{l,j+1}\Big] + \\ &2^{-j-l}\sum_{p+q=1}^{j+l-1}2^{p+q-1}\Big[(l+1)m_{pq}z_{j-p,l-q+1} - (j+1)\overline{m}_{pq}\overline{z}_{l-p,j-q+1}\Big],\end{aligned} \tag{6.3}$$

if $j+l$ is odd, and

$$\begin{aligned}(j-l)m_{jl} &= 2^{-j-l-1}\Big[(l+1)z_{j,l+1} - (j+1)\overline{z}_{l,j+1}\Big] + \\ &2^{-j-l}\sum_{p+q=1}^{j+l-1}2^{p+q-1}\Big[(l+1)m_{pq}z_{j-p,l-q+1} - (j+1)\overline{m}_{pq}\overline{z}_{l-p,j-q+1}\Big] - \\ &i\Lambda_{\frac{j+l}{2}}\psi_{jl}\end{aligned} \tag{6.4}$$

if $j+l$ is even. If $j=l$, then (6.4) implies that

$$2^{-2l}(l+1)i\,Im\Big[z_{l,l+1}+\sum_{p+q=1}^{2l-1}2^{p+q}m_{pq}z_{l-p,l-q+1}\Big]-i\Lambda_l\psi_{ll}=0.$$

From this it follows that

$$\Lambda_l=2^{-2l}(l+1)\psi_{ll}^{-1}Im\Big[z_{l,l+1}+\sum_{p+q=1}^{2l-1}2^{p+q}m_{pq}z_{l-p,l-q+1}\Big]\quad(l=1,2,3,\ldots). \tag{6.5}$$

Let us give to coefficients m_{ll}, $l=1,2,3,\ldots$, arbitrary real values. The formulas (6.3), (6.5) and (6.4) allow us to find step by step all coefficients of function $M(\overline{w},w)(\mu(x,y))$. So, from (6.3), for $j+l=1$, we find that

$$m_{10}=(z_{11}-2\overline{z}_{02})/4,\quad m_{01}=(\overline{z}_{11}-2z_{02})/4, \tag{6.6}$$

and from (6.5), for $l=1$,

$$\Lambda_1=2^{-2}\psi_{11}^{-1}Im(2z_{12}-z_{11}z_{02}). \tag{6.7}$$

Thus, formally the function $M(\overline{w},w)$ is found up to a multiplier of the form $m_{ll}\overline{w}^l w^l$ ($l=1,2,3,\ldots$). More suitable is to assume that $m_{ll}=0$, $l=1,2,3,\ldots$.

The values $\Lambda_1,\Lambda_2,\Lambda_3,\ldots$ will be called Λ values corresponding to the system of homogeneous polynomials $\{\psi_{kk}(x,y),\ k=1,2,3,\ldots\}$.

7 Relations between Λ-values and focal values

Let (6.1) be a function satisfying (6.2) with $\Lambda_k=0$ for $k=\overline{1,l-1}$ and $\Lambda_l\neq 0$. We construct a polynomial

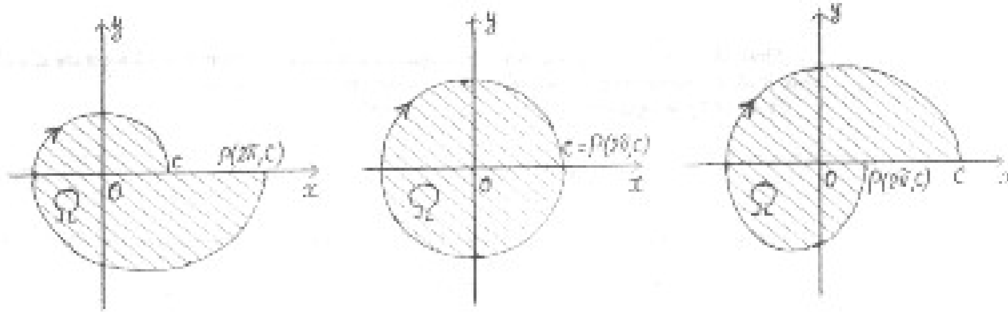
$$\tilde{\mu}(x,y)=1+\sum_{k=1}^{2l-1}\mu_k(x,y). \tag{7.1}$$

Then

$$\frac{\partial(\tilde{\mu}\tilde{P})}{\partial x}-\frac{\partial(\tilde{\mu}\tilde{Q})}{\partial y}=\Lambda_l\psi_{2l}(x,y)+\varphi_1(x,y), \tag{7.2}$$

where $\tilde{P}=y+P(x,y)$, $\tilde{Q}=x+Q(x,y)$ and $\varphi_1(x,y)$ is a holomorphic function in some neighbourhood of the origin such that $\varphi_1(x,y)=o((x^2+y^2)^l)$ as $x^2+y^2\to 0$.

Let (2.3) be a solution of the equation (2.2) with initial condition $\rho(0,c)=c$, where c is a sufficiently small positive number. We denote by Γ the closed contour consisting of the x-axis intervals $[\rho(2\pi,c),c]$ for $\rho(2\pi,c)\le c$ and $[c,\rho(2\pi,c)]$ for $\rho(2\pi,c)>c$, and the interval of trajectory $\{\rho(\varphi,c)\mid 0<\varphi<2\pi\}$. By Ω denote the closed region bounded by the contour Γ and which contains the origin (see Fig.).



By Green's formula

$$\int\limits_{\Gamma} \tilde{\mu}(\tilde{Q}dx + \tilde{P}dy) = \iint\limits_{\Omega} \Big[\frac{\partial(\tilde{\mu}\tilde{P})}{\partial x} - \frac{\partial(\tilde{\mu}\tilde{Q})}{\partial y}\Big] dxdy. \tag{7.3}$$

Let us calculate the right-hand side of equality (7.3) (see (7.2) and Fig.)

$$\iint\limits_{\Omega} \Big[\frac{\partial(\tilde{\mu}\tilde{P})}{\partial x} - \frac{\partial(\tilde{\mu}\tilde{Q})}{\partial y}\Big] dxdy = \iint\limits_{\Omega} [\Lambda_l \psi_{2l}(x,y) + \varphi_1(x,y)] dxdy =$$

$$\int\limits_0^{2\pi} d\varphi \int\limits_0^{\rho(\varphi,c)} \rho[\Lambda_l \rho^{2l} \psi_{2l}(cos\varphi, sin\varphi) + \varphi_1(\rho cos\varphi, \rho sin\varphi)] d\rho =$$

$$\pi(l+1)^{-1} \Lambda_l \psi_{2l} c^{2l+2} + \cdots .$$

Hence, by small $c > 0$, the right-hand side of (7.3) is not equal to zero (consequently, the left-hand side of (7.3) is also not equal to zero). Taking into account that integral $\int\limits_{\gamma} \tilde{\mu}(\tilde{Q}dx + \tilde{P}dy)$ vanishs along each trajectories γ of system (1.1), we have $\rho(2\pi, c) \neq c$. Let $g_k = 0$ for $k \leq m-1$ and $g_m \neq 0$. Then

$$\rho(2\pi, c) = c + g_m c^m + g_{m+1} c^{m+1} + \cdots$$

and (see Fig., (7.1), (1.2))

$$\int\limits_{\Gamma} \tilde{\mu}(\tilde{Q}dx + \tilde{P}dy) = \int\limits_{\rho(2\pi,c)}^{c} \tilde{\mu}(x,0)\tilde{Q}(x,0)dx =$$

$$\int\limits_{\rho(2\pi,c)}^{c} [(1 + \sum_{k=1}^{2l-1} \mu_k(x,0))(x + \sum_{k=2}^{\infty} Q_k(x,0))] dx = (\frac{x^2}{2} + \ldots)|_{c+g_m c^m + \ldots}^{c} =$$

$$= -g_m c^{m+1} + \cdots .$$

It follows that $m = 2l+1$ and

$$g_{2l+1} = -\pi(l+1)^{-1} \Lambda_l \psi_{2l}. \tag{7.4}$$

Note, that if $\Lambda_k = 0$, $k = \overline{1, \infty}$, then $\mu(x,y)$ (see (6.1)) is an integrating factor for (1.1) and according to [15, page 30] the system (1.1) has a centre at the origin.

Thus, the system (1.1) has in some neighbourhood of the origin $O(0,0)$ a centre if and only if all Λ-values corresponding to the system $\{\psi_{2k}(x,y)\}$ vanish.

From (6.5) we get the following centre conditions:

$$Im(z_{l,l+1} + \sum_{p+q=1}^{2l-1} 2^{p+q} m_{pq} z_{l-p,l-q+1}) = 0 \quad (l = 1, 2, 3, \dots). \tag{7.5}$$

For the cubic system (3.14) (equation (3.15)) the conditions (7.5) become (see (6.7), (6.6), (6.3)),

$$Im(2z_{12} - z_{11}z_{02}) = 0, \quad Im(2z_{20}m_{l-2,l+1} + 2z_{11}m_{l-1,l} + 2z_{02}m_{l,l-1} +$$
$$z_{30}m_{l-3,l+1} + z_{21}m_{l-2,l} + z_{03}m_{l,l-2}) = 0 \ (l = 2, 3, \dots),$$

where $m_{\beta\beta} = 0$ for every β; $m_{\alpha\beta} = 0$ for $\alpha < 0$ or $\beta < 0$;

$$m_{10} = (z_{11} - 2\bar{z}_{02})/4, \quad m_{01} = \bar{m}_{10}, \quad m_{02} = \bar{m}_{20},$$
$$m_{20} = (z_{20}\bar{z}_{11} - 2z_{20}z_{02} + z_{11}^2 - 5z_{11}\bar{z}_{02} + 6\bar{z}_{02}^2 + 2z_{21} - 6\bar{z}_{03})/32$$

and

$$\begin{aligned} m_{\alpha\beta} &= \tfrac{1}{8(\alpha-\beta)}\big[(\beta+1)(2m_{\alpha-2,\beta+1}z_{20} + 2m_{\alpha-1,\beta}z_{11} + 2m_{\alpha,\beta-1}z_{02} \\ &+ m_{\alpha-3,\beta+1}z_{30} + m_{\alpha-2,\beta}z_{21} + m_{\alpha-1,\beta-1}z_{12} + m_{\alpha,\beta-2}z_{03}) \\ &- (\alpha+1)(2m_{\alpha+1,\beta-2}\bar{z}_{20} + 2m_{\alpha,\beta-1}\bar{z}_{11} + 2m_{\alpha-1,\beta}\bar{z}_{02} \\ &+ m_{\alpha+1,\beta-3}\bar{z}_{30} + m_{\alpha,\beta-2}\bar{z}_{21} + m_{\alpha-1,\beta-1}\bar{z}_{12} + m_{\alpha-2,\beta}\bar{z}_{03})\big] \end{aligned}$$

for $\alpha + \beta = 2, 3, \dots$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha \neq \beta$.

Remark 7.1 *If we normalize the system $\{\psi_{2k}(x,y)\}$ so that*

$$\frac{1}{2\pi(k+1)} \int_0^{2\pi} \psi_{2k}(\cos\varphi, \sin\varphi)d\varphi = 1 \quad \forall k,$$

then $\Lambda_1 = \Lambda_2 = \dots = \Lambda_{m-1}$, $\Lambda_m \neq 0$ implies $V_1 = V_2 = \dots = V_{m-1} = 0$ and $V_m = \Lambda_m$.

Remark 7.2 *If for some formal series $\mu(x,y)$ we have*

$$\frac{\partial(\mu\tilde{P})}{\partial x} - \frac{\partial(\mu Q)}{\partial y} = \psi_{2l}(x,y) + \vartheta(x,y), \tag{7.6}$$

where $\vartheta(x,y) = o((x^2+y^2)^l)$ as $x^2+y^2 \to 0$ and $\psi_{2l}(x,y)$ is a polynomial of degree $2l$ ($l \geq 1$) for which (7.6) holds, then the origin is a focus for (3.14).

Remark 7.3 *If $\psi_{2l}(x,y)$ satisfies (5.7), in general, we cannot draw a conclusion concerning the existence of a focus at the origin. So, for $\mu(x,y) = 1 - xy + 3y^2$ in the case of system (5.8) we have*

$$\frac{\partial(\mu\tilde{P})}{\partial x} - \frac{\partial(\mu Q)}{\partial y} = x^2 - y^2 - x^4 + 4x^3y - 6x^2y^2 + 12xy^3 - y^4,$$

and for $\mu(x,y) = 1 + 2x + y + 8x^2 + 5xy$ in the case of system (5.9) we get

$$\begin{aligned} \tfrac{\partial(\mu\tilde{P})}{\partial x} - \tfrac{\partial(\mu Q)}{\partial y} &= x^2 - y^2 + 22x^3 - 48x^2y - 33xy^2 - 4y^3 + 16x^4 \\ &- 236x^3y - 129x^2y^2 - 20xy^3. \end{aligned}$$

For $\psi_2(x,y) = x^2 - y^2$ (5.7) is fulfilled.

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