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BIDIMENSIONAL MANIFOLDS WITH AN EXTREME PROPERTY AND THEIR EMBEDDING INTO THE THREEDIMENSIONAL MANIFOLDS

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This work is about the construction of numerable series of bidimensional manifolds with an extreme property and their embedding into the threedimensional prismatic manifolds.

1. Let the hyperbolic plane be normally and regularly decomposed by the equal regular polygons, incident by three in one top.

Maxima of the density of the Γ-regular circular packs (Γ is algebraically fixed) are achieved for this type of decomposition [1]. From the groups of the symmetries of these decompositions we extract different subgroups of the Γ without torsion so that decomposing polygons should be the fundamental domain of these subgroups. The sufficient condition of the absence of torsion in the Γ group is the existence of insignificant cycles of the vertices of the fundamental polygons [2]. Nonorientable suffices of the genus q will be obtained from the given polygons by the pairwise sides identification according to the extracted subgroups. It is evident that the surfaces obtained have a significant extreme feature - they are hyperbolic 2-manifolds (with the genus q) with the maximal circle inscribed.

The relationship between the number 2k of sides of the decomposing polygon and the genus q of the surface, given by this polygon, can be easily obtained from the Euler formula for the nonorientable surface

$$\chi = \alpha^0 - \alpha^1 + \alpha^2 = 2 - q$$

where α^i is the number oh the i-dimensional cells on the manifold $(i = 0, 1, 2)$.

2q is the number of the sides of the polygon with the canonical identification. For our surface we obtained $2k - 6(q-1)$.

2. The construction of the numerable series of the bidimensional manifolds.

Bellow we describe one of the numerable series of the bidimensional hyperbolic manifolds of the type given above (the manifolds of the given type are nonorientable).

We introduce two types of the sides identifications and we name them "elementary". They will be used for the more complicated constructions. The general scheme of the sides identification of the fundamental polygon by the consequent execution in a certain order of the elementary identifications will be given below.

Identifications α_1, α_2. The scheme of these identifications is given in figs.1,2. The sides of the polygon that are excluded from the discussion are given by the curly lines. The sides whose centres are connected on the scheme by the segment of the straight line will be identified by the shift (analog their common perpendicular). The arrow connecting the sides centres point out their identification by the sliding reflection. The vertices of the one and the same cycle are marked by the same symbols. From figs.1,2 we can see that all the vertices incident to the identification α_1 and α_2, from the cycles of the 3 vertices because of the structure of the decomposition.

The construction will be held from the genus $q > 6$.

Let us see, first of all, the even numbers $q > 6$. The construction of the series will be held only on the half of the fundamental polygon. So, we extract any of the polygon sides and we will identify them by the shift (analog their common perpendicular) with the opposite side (in fig.5 this shift is marked by the thick line). As the result, all the sides of the polygon are divided into two groups. In fact, the construction will be held only for the one of the two groups obtained (the half of the polygon). Identification of the sides groups remained can be obtained by the turning of the scheme at the angle of π (the inversion with respect of the centre of the polygon); identification α_2 is centrally symmetric, identifications α_1 are included into scheme centredly-symmetrically. For illustration, the schemes will give the identification of all the sides of the polygon. The sides identification (on the half of the polygon) will be held in the order given in brackets. When $q = 6$ it is given directly in fig.4. The scheme of such a type we write by bracket (α_1, α_2) (fig.4). For the $q = 8$ the identification is given in fig.5. We write it in such a way $(\alpha_1, \alpha_1, \alpha_2)$ (fig.5).

Let us describe the algorithm of the construction of the manifold of any genus q $(q \geq 10)$. We construct the identification from the any free edge. After this we obtain $\Delta = 3q - 10$ free sides (on the half of the polygon). But in the polygon of the manifold of the genus $q - 2$ we obtain (after the first deviding identification of the opposite sides) $\Delta' = 3q - 10$, that is equal to Δ. Free Δ sides of the polygon P_q (in such a way we will mark the fundamental polygon that corresponds to the surface of the genus q) are identified exactly according to the scheme of Δ' sides of the polygon P_{q-2}. Thus we obtain the desired reduction on q.

Let us see the odd numbers q. In this case we take arbitrary two opposite sides of the polygon and we will identify them by the screw notion.

The segment of line that connects the centres (in fig.6 it is pointed out) of these sides will be considered as an "axis" of the identification α_1 and we will construct this identification in respect to the axis (in this case we use the significant feature of the α_1 - identification). Further identifications will be held at one of the halves of the fundamental polygon. The identification of the other side can be obtained by the reflection of the scheme in respect to the axis described above.

The identification on the fundamental polygon with $q - 7 (2k - 36)$ is given in fig.6. The scheme of this identification will be given by the bracket $(i\, \alpha_1, \alpha_2)$. We fix the genus q (q is odd), $q \geq 9$. We construct the identification α_1 according to the method above. As the result, we obtain for the half of the polygon $3q - 7$ free sides. Identification α_2 absorbs (without one of boundary sides) 8 sides. Identification α_1 absorbs (without the boundary sides) 6 sides. For the genus $q = 7$ the identification is given by the bracket $(I\alpha_1, \alpha_2)$. Thus we obtain $6 + 8 = 14$ absorbed sides. In general, the $3q - 7$ free sides are identified according to the scheme (the reduction can be held in the way described above)

$$(\underbrace{I\alpha_2, I\alpha_1, \ldots, I\alpha_1}_{m}, \alpha_2),$$

but the values of $3q - 7\ (q = 7, 9, \ldots)$ and of $6m + 8\ (m = 1, 2, \ldots)$ are equal at the respective values of m and q $(q = 2m + 5)$. As the result, according to the given scheme, all the sides are identified, and all the cycles are insignificant.

3. Bidimensional manifolds of the series obtained are geodesically embedded into the threedimensional prismatic [3] manifold. As it was done in [3], we will construct a right prism above the $2k-$ polygon with the angle $\frac{\pi}{2}$ at the lower foundation and at the upper one the angle is $\frac{2\pi}{12}$ (fig. 7). Reflect the prism obtained in respect to $2k-$ polygon. $2k-$ polygon we name the middle section of the prism obtained. Lateral faces of the prism will be identified according to the middle section sides identification. That is, the lateral faces, that correspond to the sides of the middle section, identified by the shift, we will also identify by the shift (at the same vector). The faces corresponding to the sides identified by the screw motion, will be identified by the stepwise execution of the screw motion and reflection in respect to the middle section. The upper foundation is identified with the lower one by the screw motion with the axis along the common perpendicular of the foundations of the prism with angle of the tort $2\pi/l\ (l = 2k = 6(q - 1))$. It is clear that due to our constrictions every side rib is included in the insignificant cycle that corresponds to the cycle of the

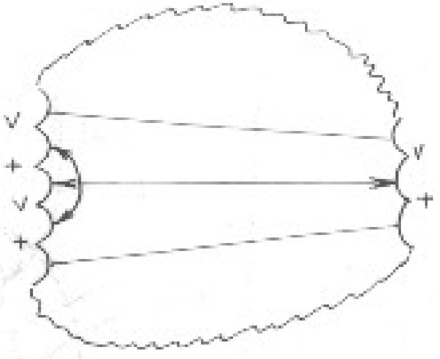

Fig.1. α_1.

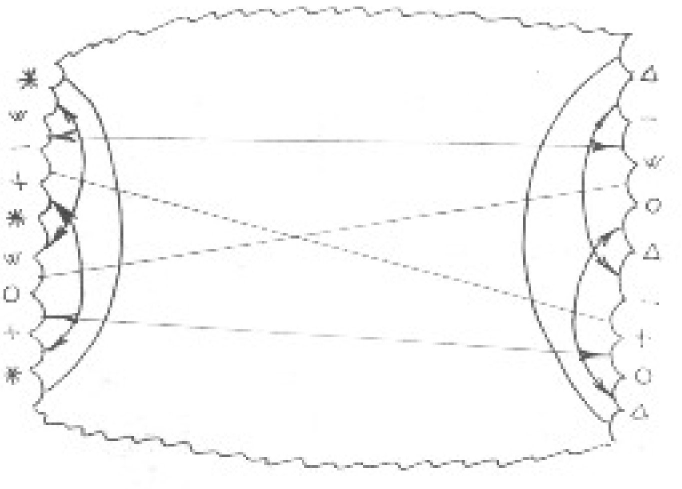

Fig.2. α_2.

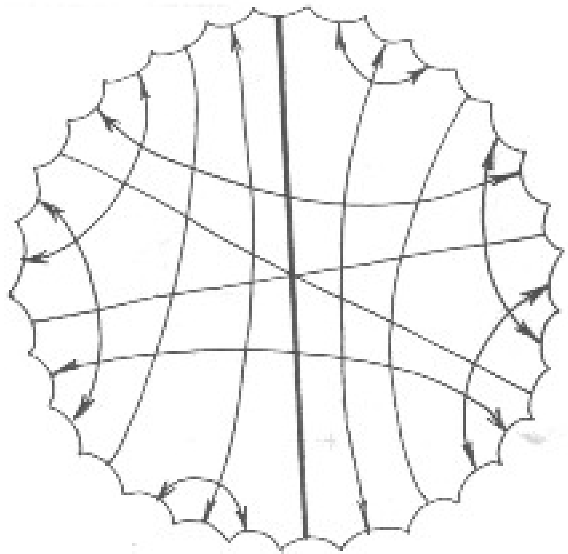

Fig. 4. q = 6, 2k = 30

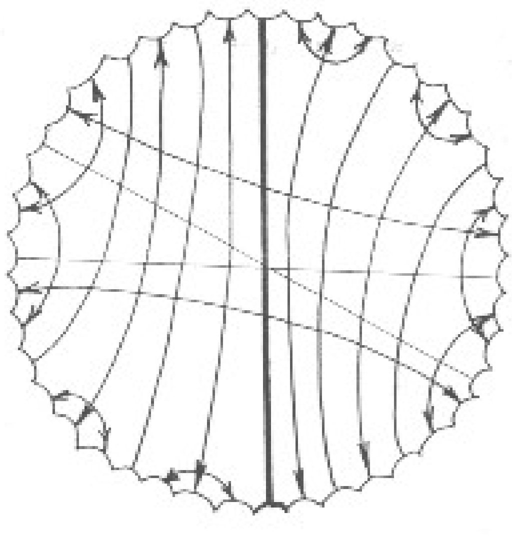

Fig. 5. q = 8, 2k = 42

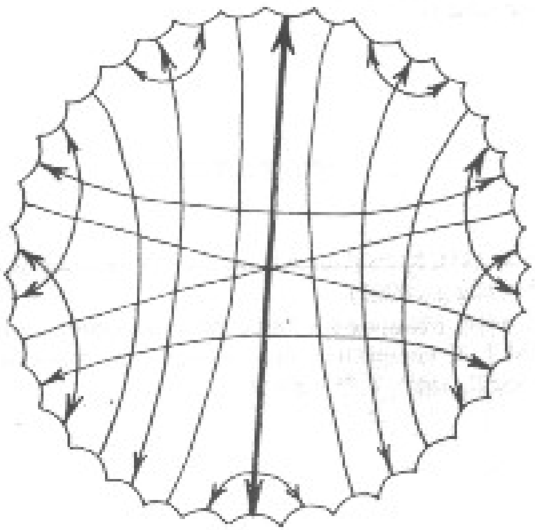

Fig. 6. q = 7, 2k = 36

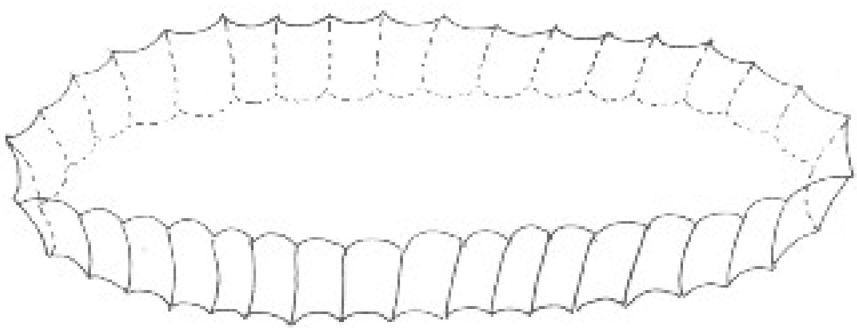

Fig. 7.

vertex. Not complicated but awkward direct calculation (according to the [3]) shows that also all the ribs incident to the foundations are included in the insignificant cycles each collecting 12 ribs.

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