Bul. Ştiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XIV(1998), Nr. 1, 5-14

Dedicated to Professor Iuliu Coroian on his 60th anniversary

SOME HERMITE BIVARIATE INTERPOLATION PROCEDURES

Dan BĂRBOSU

Abstract. A blending operator of Hermite type is constructed by using the parametric extensions of the univariate Hermite projectors. These extensions are presented in the first section.

In the next section, the product of these extensions is considered. It is proved that this product is an interpolation projector and its precision set is determinated.

The main results of the papes are contained in the last section, where is proved that the boolean sum operator of Hermite parametric extensions is an interpolation projector (theorem 4.1), is determined the precision set of this projector (theorem 4.2) and are deduced the expressions of the corresponding remainder operator (theorem 4.3 and theorem 4.4).

1991 Mathematics Subject Classification: 41A05, 41A15, 41A35
Key words and phrases: Hermite projector, parametric extension, boolean sum, precision set, blending operator.

1. Introduction.

The notions of "parametric extensions" and "boolean sum operator" were introduced by W. J.Gordon [4]. The "boolean sum operator" allows us to construct the so called "blending operator".

The object of this note is to construct some blending operators of Hermite type. The expressions of the blending approximants, the precision sets and the remainders associated to these approximants are determined.

2. Parametric extensions of the Hermite projectors

Let a,b,c,d be real numbers so that $a<b$ and $c<d$. If m,n are positive integers, one denotes by $x_k \in [a,b]$, $k=0,1,\ldots,m$ the knots satisfying the conditions $x_i \neq x_s$ for any $i,s \in \{0,1,\ldots,m\}$ $i \neq s$, and by $y_j \in [c,d]$, $j=0,1,\ldots,n$ the knots satisfying the conditions $y_p \neq y_q$ for any $p,q \in \{0,1,\ldots,n\}$, $p \neq q$.

If $r_0, r_1, \ldots, r_m$, $\bar{r}_0, \bar{r}_1, \ldots, \bar{r}_n$ are nonnegative integers, one defines the integers m_1, n_1 by the following equalities

(2.1) $$m_1 = m + r_0 + r_1 + \ldots + r_m$$

(2.2) $$n_1 = n + \bar{r}_0 + \bar{r}_1 + \ldots + \bar{r}_n$$

Next, one considers the spaces of functions X and Y, defined respectively by

(2.3) $$X = \{ f \in C^{m_1-1}[a,b] \mid (\exists) f^{(m_1)} \text{ on } [a,b] \},$$

(2.4) $$Y = \{ g \in C^{n_1-1}[c,d] \mid (\exists) g^{(n_1)} \text{ on } [c,d] \}.$$

The Hermite projector $H_{m_1} : X \to X$ associates to any function $f \in X$ the approximant $H_{m_1}(f)$ given by

(2.5) $$H_{m_1}(f)(x) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(x) f^{(j)}(x_k)$$

In (2.5), $h_{kj}(x)$ denotes the fundamental Hermite polynomials, i.e.

(2.6) $$h_{kj}(x) = \frac{(x-x_k)^j}{j!} u_k(x) \sum_{\nu=0}^{r_k-j} \frac{(x-x_k)^\nu}{\nu!} \left[\frac{1}{u_k(x)} \right]^{(\nu)}_{x=x_k}$$

with

(2.7) $$u_k(x) = \frac{u(x)}{(x-x_k)^{r_k+1}}, \quad u(x) = \prod_{i=0}^{m}(x-x_i)^{r_i+1}$$

In a similar way, the Hermite projector $\overline{H}_{n_1} : Y \to Y$ associates to any function $g \in Y$ the approximant $\overline{H}_{n_1}(x)$, defined by

(2.8) $$\overline{H}_{n_1}(g)(y) = \sum_{i=0}^{n}\sum_{p=0}^{\bar{r}_i} \bar{h}_{ip}(y)\, g^{(p)}(y_i)$$

$\bar{h}_{ip}(y)$ being the fundamental Hermite polynomials.

Let f be a bivariate function, $f \in C^{m_1,n_1}([a,b]\times[c,d])$. The parametric extensions of the projectors $H_{m_1}, \overline{H}_{n_1}$ are defined respectively by

(2.9) $$H^x_{m_1}(f)(x,y) = \sum_{k=0}^{m}\sum_{j=0}^{r_k} h_{kj}(x)\, f^{(j,0)}(x_k,y)$$

(2.10) $$\overline{H}^y_{n_1}(f)(x,y) = \sum_{i=0}^{n}\sum_{p=0}^{\bar{r}_i} \bar{h}_{ip}(y)\, f^{(0,p)}(x,y_i)$$

Theorem 2.1. *The operators $H^x_{m_1}, \overline{H}^y_{n_1}$ given at (2.9), (2.10) are interpolation projectors.*

Proof. Taking into account that $H_{m_1}, \overline{H}_{n_1}$ are interpolation projectors, it results (see [3]) that their parametric extensions are interpolation projectors.

Theorem 2.2. *The precision sets of $H^x_{m_1}, \overline{H}^y_{n_1}$ are given respectively by*

(2.11) $$\mathcal{P}(H^x_{m_1}) = \{\underbrace{x_0,\dots,x_0}_{r_0+1},\dots,\underbrace{x_m,\dots,x_m}_{r_m+1}\}\times[c,d]$$

and

(2.12) $$P(\overline{H}^y_{n_1}) = [a,b] \times \{\underbrace{y_0,\dots,y_0}_{\bar r_0+1},\dots,\underbrace{y_n,\dots,y_n}_{\bar r_n+1}\}$$

Proof. The precision sets of the univariate projectors $H_{m_1}, \overline{H}_{n_1}$ are

$$P(H_{m_1}) = \{\underbrace{x_0,\dots,x_0}_{r_0+1},\dots,\underbrace{x_m,\dots,x_m}_{r_m+1}\}$$

$$P(\overline{H}_{n_1}) = \{\underbrace{y_0,\dots,y_0}_{\bar r_0+1},\dots,\underbrace{y_n,\dots,y_n}_{\bar r_n+1}\}$$

Applying a well known result due to F.J.Devols and W.Schemp [3], one obtain (2.11) and (2.12).

Remark 2.1. From the theorem 2.2 it results that $H^x_{m_1}, \overline{H}^y_{n_1}$ have the following interpolatory properties

(2.13) $$\left(H^x_{m_1}(f)\right)^{(j,0)}(x_k,y) = f^{(j,0)}(x_k,y), \quad k=\overline{0,m}, \; j=\overline{0,r_k}$$

(2.14) $$\left(\overline{H}^y_{n_1}(f)\right)^{(0,p)}(x,y_i) = f^{(0,p)}(x,y_i), \quad i=\overline{0,n}, \; p=\overline{0,\bar r_i}$$

where $x\in[a,b]$, $y\in[c,d]$.

Remark 2.2. If $r_0=r_1=\dots=r_m=0$, $\bar r_1=\bar r_2=\dots=\bar r_n=0$, the projector $H_{m_1}, \overline{H}_{n_1}$ are reduced to the univariate Lagrange projectors $L_m, \overline{L}_n$.

For more details, see [1].

3. The product of Hermite parametric extensions

Let $D^{m_1,n_1}([a,b]\times[c,d])$ be the following space of functions

(3.1) $D^{m_1,n_1}([a,b]\times[c,d]) =$

$$= \left\{ f\in C^{m_1,n_1}([a,b]\times[c,d]) \;\middle|\; \frac{\partial^{j+p}f}{\partial x^j\partial y^p}(x,y) = \frac{\partial^{p+j}f}{\partial y^p\partial x^j}(x,y),\; j=\overline{0,m_1},\; p=\overline{0,n_1} \right\}$$

Clearly, $D^{m_1,n_1}([a,b]\times[c,d])$ is a subspace of the linear space $C^{m_1,n_1}([a,b]\times[c,d])$.

Theorem 3.1. *The projectors*

$H^x_{m_1}, \overline{H}^y_{n_1} : D^{m_1,n_1}([a,b]\times[c,d]) \to D^{m_1,n_1}([a,b]\times[c,d])$ *commute.*

Their product $H^x_{m_1}\overline{H}^y_{n_1}$ *is an interpolation projector.*

Proof. The first part results by direct computation. For the second part, the general result from [3] is applied.

Remark 3.1. The product projector $H^x_{m_1}\overline{H}^y_{n_1}$ associates to any $f\in D^{m_1,n_1}([a,b]\times[c,d])$ the approximant $H^x_{m_1}\overline{H}^y_{n_1}(f)$, defined by

$$(3.2)\qquad H^x_{m_1}\overline{H}^y_{n_1}(f)(x,y)=\sum_{k=0}^{m}\sum_{j=0}^{r_k}\sum_{i=0}^{n}\sum_{p=0}^{\bar r_i} h_{kj}(x)\,\bar h_{ip}(y)\,f^{(j,p)}(x_k,y_i)$$

Theorem 3.2. *The product interpolation projector* $H^x_{m_1}\overline{H}^y_{n_1}$ *has the precision set* $\mathcal{P}(H^x_{m_1},\overline{H}^y_{n_1})$, *expressed by*

$$(3.3)\qquad \mathcal{P}(H^x_{m_1},\overline{H}^y_{n_1})=\{\underbrace{x_0,\dots,x_0}_{r_0+1},\dots,\underbrace{x_m,\dots,x_m}_{r_m+1}\}\times\{\underbrace{y_0,\dots,y_0}_{\bar r_0+1},\dots,\underbrace{y_n,\dots,y_n}_{\bar r_n+1}\}$$

Proof. Taking into account the expressions of $\mathcal{P}(H^x_{m_1})$, $\mathcal{P}(\overline{H}^y_{n_1})$ and the general result [3] $\mathcal{P}(H^x_{m_1}\overline{H}^y_{n_1})=\mathcal{P}(H^x_{m_1})\cap\mathcal{P}(\overline{H}^y_{n_1})$, it follows (3.3).

Remark 3.2. From the theorem 3.2, it results that the product projector $H^x_{m_1}\overline{H}^y_{n_1}$ has the following interpolatory properties

$$\left((H^x_{m_1}\overline{H}^y_{n_1})(f)\right)^{(i,p)}(x_k,y_j) = f^{(i,p)}(x_k,y_j) \tag{3.4}$$

for $k=\overline{0,m}$, $i=\overline{0,r_k}$, $j=\overline{0,n}$, $p=\overline{0,\bar{r}_j}$.

Remark 3.3. In the particular case $r_0=r_1=\dots=r_m=0$; $\bar{r}_0=\bar{r}_1=\dots=\bar{r}_n=0$, the Hermite product projector $H^x_{m_1}\overline{H}^y_{n_1}$ reduces to Lagrange product projector $L^x_m\overline{L}^y_n$. In the particular case $m=0$, $n=0$ one has $m_1=r_0$, $n_1=\bar{r}_0$ and the univariate Hermite projectors are reduced to the parametric extensions method, on generate the Taylor product projector $T^x_m\overline{T}^y_n$. For more details, see our paper [1].

4. The boolean sum of the Hermite parametric extensions

The boolean sum of the Hermite parametric extensions is defined by the following equality

$$H^x_{m_1}\oplus\overline{H}^y_{n_1} = H^x_{m_1}+\overline{H}^y_{n_1} - H^x_{m_1}\overline{H}^y_{n_1} \tag{4.1}$$

In this section some approximation properties of $H^x_{m_1}\oplus\overline{H}^y_{n_1}$ will be estabilished.

Theorem 4.1. $H^x_{m_1}\oplus\overline{H}^y_{n_1}$ *is an interpolation projector on* $D^{m_1,n_1}([a,b]\times[c,d])$.

Proof. It is well known (see [3]) that the boolean sum of two interpolation projectors is an interpolation projector. Via the theorem 2.1, it follows the conclusion of the theorem 4.1.

Theorem 4.2. The precision set $\mathcal{P}(H_m^x \oplus \overline{H}_n^y)$ is given by

$$(4.2)\quad \mathcal{P}(H_{m_1}^x \oplus \overline{H}_{n_1}^y) = (\{x_0,\ldots,x_0,\ldots,x_m,\ldots,x_m\} \times [c,d]) \cup \cup ([a,b] \times \{y_0,\ldots,y_0,\ldots,y_n,\ldots,y_n\})$$

Proof. The precision set of the boolean sum (see [3]) $H_{m_1}^x \oplus \overline{H}_{n_1}^y$ is

$$(4.3)\quad \mathcal{P}(H_{m_1}^x \oplus \overline{H}_n^y) = \mathcal{P}(H_{m_1}^x) \cup \mathcal{P}(\overline{H}_{n_1}^y)$$

From the theorem 2.2, taking into account (4.3), it results that (4.2) holds.

Remark 4.1. The projector $H_{m_1}^x \oplus \overline{H}_n^y$ associates to any function $f \in D^{m_1,n_1}([a,b] \times [c,d])$ the approximant $H_{m_1}^x \oplus \overline{H}_{n_1}^y(f)$, given by

$$(4.4)\quad H_{m_1}^x \oplus \overline{H}_{n_1}^y(f)(x,y) = \sum_{k=0}^{m} \sum_{i=0}^{n} \sum_{j=0}^{r_k} \sum_{p=0}^{\bar{r}_i} [f^{(j,0)}(x_k,y) + f^{(0,p)}(x,y_i) - f^{(j,p)}(x_k,y_i)]\, h_{kj}(x)\, \bar{h}_{ip}(y)$$

From the theorem 4.2, it results that the projector $H_m^x \oplus \overline{H}_n^y$ has the following interpolation properties

$$(4.5)\quad \left(H_{m_1}^x \oplus \overline{H}_n^y(f)\right)^{(j,p)}(x_k,y) = f^{(j,p)}(x_k,y)$$

$$(4.6)\quad \left(H_{m_1}^x \oplus \overline{H}_{n_1}^y(f)\right)^{(j,p)}(x,y_i) = f^{(j,p)}(x,y_i)$$

where $k=\overline{0,m}$, $j=\overline{0,r_k}$, $i=\overline{0,n}$, $p=\overline{0,\bar{r}_i}$, $x \in [a,b]$, $y \in [c,d]$.

The projector $H_{m_1}^x \oplus \overline{H}_{n_1}^y$ is called "blending operator of Hermite type".

Next, the blending interpolation formula

$$f = H^x_{m_1} \oplus \overline{H}^y_{n_1}(f) + R^{xy}_{m_1 n_1}(f) \tag{4.7}$$

will be considered and the remainder term $R^{xy}_{m_1 n_1}(f)$ will be studied.

Theorem 4.3. *Let be* $\alpha_1 = \min\{x, x_0, \ldots, x_m\}$, $\beta_1 = \max\{x, x_0, \ldots, x_m\}$, $\alpha_2 = \min\{y, y_0, \ldots, y_n\}$, $\beta_2 = \max\{y, y_0, \ldots, y_n\}$. *If* $f \in D^{m_1, n_1}([a,b] \times [c,d])$ *and there exists* $f^{(m_1+1, n_1+1)}$ *on* $]\alpha_1, \beta_1[\times]\alpha_2, \beta_2[$, *then there exists a point* $(\xi, \mu) \in]\alpha_1, \beta_1[\times]\alpha_2, \beta_2[$ *so that one has*

$$R^{xy}_{m_1 n_1}(f)(x,y) = \frac{u(x) \cdot \overline{u}(y)}{(m_1+1)!\,(n_1+1)!} f^{(m_1+1, n_1+1)}(\xi, \mu) \tag{4.8}$$

Proof. The remainder operator associated to the boolean sum projector $H^x_{m_1} \oplus \overline{H}^y_{n_1}$ is the product $R^x_{m_1} \overline{R}^y_{n_1}$ of the remainders associated to $H^x_{m_1}$ and $\overline{H}^y_{n_1}$. Taking into account that

$$R^x_{m_1}(f)(x,y) = \frac{u(x)}{(m_1+1)!} f^{(m_1+1, 0)}(\xi, y), \quad \xi \in]\alpha_1, \beta_1[\tag{4.9}$$

and

$$\overline{R}^y_{n_1}(f)(x,y) = \frac{\overline{u}(y)}{(n_1+1)!} f^{(0, n_1+1)}(x, \mu), \quad \mu \in]\alpha_2, \beta_2[\tag{4.10}$$

it follows that (4.8) is true.

Theorem 4.4. *If* $f \in D^{m_1+1, n_1+1}([a,b] \times [c,d])$, *then*

$$R^{xy}_{m_1 n_1}(f)(x,y) = \int_a^b \int_c^d \varphi_{m_1}(x;s)\, \overline{\varphi}_{n_1}(y;t)\, f^{(m_1+1, n_1+1)}(s,t)\, ds\, dt \tag{4.11}$$

where

$$(4.12)\qquad \varphi_{m_1}(x;s)=\frac{1}{m_1!}\left\{(x-s)_+^{m_1}-\sum_{k=0}^{m}\sum_{j=0}^{r_k}h_{kj}(x)\left[(x_k-s)_+^{m_1}\right]^{(j)}\right\}$$

$$(4.13)\qquad \overline{\varphi}_{n_1}(y;t)=\frac{1}{n_1!}\left\{(y-t)_+^{n_1}-\sum_{i=0}^{n}\sum_{p=0}^{r_i}h_{ip}(y)\left[(y_i-t)_+^{n_1}\right]^{(p)}\right\}$$

Proof. By using the Peano theorem, it results that the remainders associated to $R_{m_1}^x$, $\overline{R}_{n_1}^y$ are respectively

$$(4.14)\qquad R_{m_1}^x(f)(x,y)=\int_a^b \varphi_{m_1}(x;s)\,f^{(m_1+1,0)}(s,y)\,ds$$

and

$$(4.15)\qquad \overline{R}_{n_1}^y(f)(x,y)=\int_c^d \overline{\varphi}_{n_1}(y;t)\,f^{(0,n_1+1)}(x,t)\,dt$$

Taking into account that $R_{m_1 n_1}^{xy}=R_{m_1}^x\overline{R}_{n_1}^y$, it follows that (4.11) holds.

Finally, one remark that the Hermite blending interpolation formula (4.7) contains as particular cases the blending Lagrange interpolation formula and the blending Taylor interpolation formula. More details about these formulas are contained in our paper [1].

REFERENCES

1. **Bărbosu, D.**, Approximation of multivariate functions by using boolean sums of positive linear operators of interpolatory type, Ph. D. Thesis, Cluj-Napoca, 1997
2. **Coman, Gh.**, Numerical Analysis, editura LIBRIS, Cluj-Napoca, 1995(in Romanian)
3. **Delvos, F. J., Schemp, W.**, Boolean Methods in Interpolation and Approximation, Longman Scientific & Technical, 1989
4. **Gordon W.J.**, Distributive Lattices and the Approximation of Multivariate Functions, in "Aprox. With Sfer. Emphasis on Spline Functions", Acad. Press, New-Jork (1962), 223-277

Received 15.06.1998

North University of Baia Mare
Victoriei 76, Baia Mare 4800
ROMANIA
E-mail:dbarbosu@univer.ubm.ro