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Dedicated to Professor Iulian Coroian on his 60th anniversary

THE CONNECTION BETWEEN THE RELATIVISTIC EQUATION OF THE HYDROSTATICAL BALANCE AND THE U^0 COMPONENT OF THE SPACE-TIME QUADRI VELOCITY

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Abstract. In this paper a connection is established between the equation of the hydrostatical balance and the variation of the u^0 component inside a massive body with a spherical symmetry - where the relativistic effects are not negligible.

At the mathematical modelling of the space-time universe we have to start from metrics of a Riemann-space with 4 dimensions

(1) $$ds^2 = g_{ik}dx^i dx^k =$$

$$= g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2(g_{01}dx^0dx^1 + g_{02}dx^0dx^2 +$$

(1') $$+ g_{03}dx^0dx^3 + g_{12}dx^1dx^2 + g_{13}dx^1dx^3 + g_{23}dx^2dx^3)$$

- g_{ik} - is called the fundamental metric tensor and it is a covariant symmetric tensor, that is

$$g_{\mu\nu} = g_{\nu\mu} \qquad (2)$$

- x^0 - is the temporal coordinate
- x^1, x^2, x^3 - the spacial coordinates measured by a distant observer

If we consider the cartesian frame O_{xyz} starting from the center of a massive body with a spherical symmetry (Fig.1)

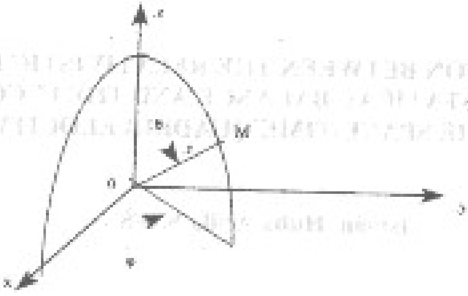

Fig.1

so

$$x^0 = t = ct_{ph} ;\ x^1 = r ,\ x^2 = \theta ,\ x^3 = \varphi \quad \text{(spherical coordinates)} \qquad (4)$$

The "ph" index denotes the quantities measured in physical unite (c.g.s) and without index it is the same quantities measured in geometrical units, that is the corresponding physical quantities multiplied by the universal constants

$$\begin{cases} c - \textit{speed of light in vacue} \\ G - \textit{gravitational constant} \\ h - \textit{Planck's constant} \left(\textit{or } \hbar = \dfrac{h}{2\pi} \right) \end{cases} \qquad (5)$$

Therefore:

$$r = r_{ph}\ (cm)\quad ;\quad t = ct_{ph}\ (cm)$$

$$\psi = \psi_{ph}\ (cm^{0})\ ;\quad \theta = \theta_{ph}\ (cm^{0})$$

linear speed: $$v = \frac{v_{ph}}{c}\ (cm^{0})$$

mass: $$m = \frac{G}{c^{2}} M_{ph}\ (cm) \tag{6}$$

acceleration: $$a = \frac{1}{c^{2}} a_{ph}\ (cm^{-1})$$

pressure: $$P = \frac{G}{c^{4}} P_{ph}\ (cm^{-2})$$

energy density: $$\varepsilon = G \cdot \rho_{ph}\ (cm^{-2}),\ (\rho_{ph} = density)$$

If the central mass has a spherical symmetry, "O" being the center of the symmetry, the metrics will be

$$ds^{2} = g_{00}(dx^{0})^{2} + g_{11}(dx^{1})^{2} + g_{22}(dx^{2})^{2} + g_{33}(dx^{3})^{2} \tag{7}$$

For knowing the space-time, we'll define the QUADRI VELOCITY

$$u(u^{0}, u^{1}, u^{2}; u^{3}), \tag{8}$$

with the components:

$$u^{i} = \frac{dx^{i}}{ds} \tag{9}$$

in our case

$$u(u^{0}, 0, 0, 0) \tag{10}$$

because

$$u^{\alpha} = \frac{dx^{\alpha}}{ds} = \frac{dx^{\alpha}}{dx^{0}} \cdot \frac{dx^{0}}{ds} = 0 \cdot u^{0} = 0,\quad \alpha \in \{1, 2, 3\} \tag{11}$$

From metrics (7) $\left(\frac{dx}{dx^0}\right)^2 = g_{00} = \left(\frac{1}{u^0}\right)^2$; or

$$u^0 = g_{00}^{-\frac{1}{2}} \tag{12}$$

The contra variant components of the energy-impulse tensor can be defined as follows:

$$T^{ik} = (\varepsilon + P) u^i u^k - P g^{ik} \; ; \; T^{ik} = T^{ki} \tag{13}$$

where

$$g_{ik} g^{mi} = \delta_k^m \tag{14}$$

$$\delta_k^m = Kronecker\text{'s}\,symbol = \begin{cases} 1 \; , \; if \; k = m \\ 0 \; ; \; if \; k \neq m \end{cases}$$

Therefore

$$g^{ii} = \begin{cases} \dfrac{1}{g_{ii}} \; ; \; k = m \\ 0 \; ; \; i \neq 0 \end{cases} \tag{15}$$

The mixed components of the energy-impulse tensor are

$$T_r^i = T^{ik} g_{kr} = (\varepsilon + P) u^i u^k g_{kr} - P g^{ik} g_{kr} \tag{16}$$

In our case

$$T_0^0 = \varepsilon \; ; \; T_1^1 = T_2^2 = T_3^3 = -P \; , \tag{17}$$

the other components being null.

The equations of the hydrostatic balance express Einstein's equation, in derivates form (Ionescu Pallas, 1980; Zeldovich and Novikov, 1971),

$$T_{i,k}^k + \Gamma_{mk}^k T_i^m - \Gamma_{ik}^m T_m^k = 0 \tag{18}$$

In the tensorial calculus f_i represent the partial derivatives of function f with respect to x^i.

$$\Gamma_{kl}^l = \frac{1}{2} g^{lm} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right) \tag{19}$$

they respect Christoffel's symbols of the second kind. For the spherical symmetry the following components will be considered non-null

$$\Gamma^0_{01} = \frac{g_{00,1}}{2g_{00}}; \quad \Gamma^1_{00} = -\frac{g_{00,1}}{2g_{11}}; \quad \Gamma^1_{11} = \frac{g_{11,1}}{2g_{11}}$$

$$\Gamma^1_{22} = -\frac{g_{22,1}}{2g_{11}}; \quad \Gamma^1_{33} = -\frac{g_{33,1}}{2g_{11}}; \quad \Gamma^2_{12} = \frac{g_{22,1}}{2g_{22}} \tag{20}$$

$$\Gamma^2_{33} = -\frac{g_{33,2}}{2g_{22}}; \quad \Gamma^3_{13} = \frac{g_{33,1}}{2g_{33}}; \quad \Gamma^3_{23} = \frac{g_{33,2}}{2g_{33}}$$

After the suitable replacement in the relation (18), taking in turn ($k = 0, 1, 2, 3$), we'll get

$$T^0_{0,0} = 0; \quad T^1_{1,1} + (P + \varepsilon)\Gamma^0_{10} = 0; \quad T^2_{2,2} = 0; \quad T^3_{3,3} = 0 \tag{21}$$

or

$$\frac{\partial \varepsilon}{\partial t} = 0; \quad \frac{\partial P}{\partial r} = -(P + \varepsilon)\frac{g_{00,1}}{2g_{00}}; \quad \frac{\partial P}{\partial \theta} = 0; \quad \frac{\partial P}{\partial \varphi} = 0 \tag{22}$$

The first relation shows us that ε does not depend on time. The last two relations tells us that the pressure doesn't depend on θ and φ; that result from the spherical symmetry. Whereas

$$\frac{g_{00,1}}{2g_{00}} = \frac{\partial}{\partial r}\ln g_{00}^{\frac{1}{2}} = \frac{\partial(\ln u^0)}{\partial r} = \frac{d(\ln u^0)}{dr} \tag{23}$$

we'll get, finally

$$\frac{dP}{P + \varepsilon} = -d(\ln u^0) \tag{24}$$

which transcribed in physical values becomes the equation of the relativistic hydrostatic balance inside a spherical body.

$$\frac{dP}{dr}=\frac{-G\left(\rho+\frac{P}{c^2}\right)\left[M(r)+\frac{4\pi}{c^2}Pr^3\right]}{r^2\left[1-\frac{2GM(r)}{c^2r}\right]} \tag{25}$$

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