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Dedicated to Professor Iulian Coroian on his 60[th] anniversary

NONLINEAR PROGRAMMING FOR SET VALUED EXTREMAL SOLUTION OF LINEAR AND ON NONLINEAR LARGE SYSTEMS OF INEQUALITIES

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Abstract. We show that certain generalized Newton-type methods may be extended and applied useful for numerical solving of large systems of nonlinear inequalities of the following form.

Find $\bar{x}$ such that

(1.1) $$\min f(\bar{x}) = \Theta \quad g(\bar{x}) \leq \Theta \quad and \quad f(\bar{x}) \in K \subset Y, \qquad \bar{x} \in X$$

where $f : X \to Y$; X *and* Y being reflexive Banach spaces, and K represents a certain semiordered nonempty closed cone in Banach space Y.

We suppose the convexity of $f(x)$ and $g(x)$ respectively. In the next we shall study the convex optimization problem using the methods

(1.3) $$x_n = x_{n-1} - \frac{f(x_{n-1})}{[x_{n-1}, x_{n-1} + g_{n-1}]} g_{n-1}$$

or

(1.4) $$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1}) g_{n-1}} g_{n-1}$$

Here we have applied the of course the I. order difference quotient instead of the derivative $F'(x_{n-1})$.

1. The present paper is divided in following paragraphs:

§1 Introduction. Monotone convergence of the above methods. (1.3) and (1.4)

§2 Proportional distribution of a given amount of money among companies considering the efficiency.

§3 Nonlinear singular first-order system of differential equations with bilocal conditions.

§4 Optimal control.

§5 Inventory problems.

§6 A certain physical numerical illustration. Optimality of a given parallelepiped.

§7 The problem of vertebrate retina.

§8 Large nonlinear dynamical systems. Global problems.

1. Introduction. Monotone convergence of the methods, (1.3 and 1.4)

For the next we consider the following notations: let X be a given Riesz-space endowed with the scalar product (x,x). Moreover consider the continuous functional $F : X \to R$, where R denotes the set of the real numbers. Let us assume that we have already constructed the first-order difference quotient i.e. $F_{z_1 z_2}$ on the knots z_1, z_2 and which is additive.

(A) $$F_{z_1 z_2}(z_1 + z_2) = F_{z_1 z_2}(z_1) + F_{z_1 z_2}(z_2)$$
for the arbitrary elements z_1, z_2, where

$$z_1 = x_n, z_2 = x_{n-1} \in X, \quad (n = 1, 2, \dots)$$
furthermore the next equality holds

(B) $$F_{x_n x_{n-1}}(x_n - x_{n-1}) = F(x_n) - F(x_{n-1})$$
We define the second-order difference quotient as usual, starting with

$$\tilde{F}(x) := F_{x z_1}$$

and $\tilde{F}_{z_2 z_3}$. The last difference quotient is additive and fulfils

$$\tilde{F}_{z_2 z_3}(z_2 - z_3) = \tilde{F}(z_2) - \tilde{F}(z_3)$$
i.e.

$$F_{z_1 z_2 z_3}(z_2 - z_3) = F_{z_1 z_2} - F_{z_1 z_3}$$
so we get the formula

(1.2) $$F(x_n) = F(x_{n-1}) + F_{x_{n-1} x_{n-2}}(x_n - x_{n-1}) +$$

$$+ F_{x_n x_{n-1} x_{n-2}}(x_{n-1} - x_{n-2})(x_n - x_{n-1})$$
Let us construct the next iterative method in the non differentiable case

(1.3) $$x_n = x_{n-1} - \frac{F(x_{n-1})}{F_{x_{n-1} x_{n-2}} y_{n-1}} y_{n-1}.$$
Here, it is applied the I. order difference quotient instead of the derivative $F'(x_{n-1})$. The approximate elements $y_{n-1} \in X, (n = 1, 2, \dots)$ will be chosen later.

(1.4) $$x_n = x_{n-1} - \frac{F(x_{n-1})}{F'(x_{n-1}) y_{n-1}} y_{n-1}$$
We should mention, that the inverse of $F_{x_n x_{n-1}}$ does not appear but the reciprocal. For the construction of divided differences see the examples giving [15-18]. We consider the functional

$$F(x) = f(\xi_1, \xi_2, \dots, \xi_m) \in R.$$

where

$$x = (\xi_1, \xi_2, \dots, \xi_m);$$

and let the nodes

$$x_1 = (u_1, u_2, \dots, u_m),\ x_2 = (v_1, v_2, \dots, v_m),\ x_3 = (w_1, w_2, \dots, w_m).$$

Then we can construct the first and second partial divided differential in the usual way:

$$f(\dots u_i v_i \dots) = \frac{f(\dots u_i \dots) - f(\dots v_i \dots)}{u_i - v_i}$$

$$f(\dots u_i v_i w_i \dots) := \frac{f(\dots u_i v_i \dots) - f(\dots v_i w_i \dots)}{u_i - w_i},$$

$$P^{(1)}_{x_1 x_2} \dots = \Big\{ f(u_1 v_1; u_2, \dots, u_m), \dots f(v_1; u_2 v_2; u_3, \dots, u_m), \dots;$$

$$; f(v_1, v_2; u_3 v_3; u_4, \dots, u_m), \dots;$$

$$; f(v_1, v_2, \dots, v_{m-1}; u_m v_m), \dots \Big\}$$

Let assume that originally is given the operator equation $P(x) = \Theta$ where $P, U :$

$X \longrightarrow X$ and X is a Riesz-space endowed with a certain scalar product, and the operator U shall be choosed in the next in a suitable manner. Let be

(1.5) $$F(x) := \Big\langle P(x), U(x) \Big\rangle \Longleftrightarrow F(x) = 0 \Longleftrightarrow P(x) = \Theta$$

In this case, the approximate element y_n can be chosen as follows: starting with expression

$$F_{x_{n-1} x_{n-2}} \dots = \Big\langle P(x), U(x) \Big\rangle_{\dots} =$$

$$= \Big\langle P_{x_{n-1} x_{n-2}} \dots, U(x_{n-1}) \Big\rangle + \Big\langle P(x_{n-2}), U_{x_{n-1} x_{n-2}} \dots \Big\rangle =$$

$$= \Big\langle P_{x_{n-1} x_{n-2}} \dots, U(x_{n-1}) + P(x_{n-2}) \Big\rangle =$$

$$= \Big\langle \dots, P^T_{x_{n-1} x_{n-2}} \Big(U(x_{n-1}) + P(x_{n-2}) \Big) \Big\rangle$$

and using the notation:

$$Q(x_{n-1}, x_{n-2}, U) := P^T_{x_{n-1} x_{n-2}} \Big(U(x_{n-1}) + P(x_{n-2}) \Big),$$

(where $P^T_{x_{n-1} x_{n-2}}$ denotes the transposed of $P_{x_{n-1} x_{n-2}}$), we get

(1.6) $$g_{n-1} = \frac{Q(x_{n-1}, x_{n-2}, \Gamma)}{\left\|Q(x_{n-1}, x_{n-2}, \Gamma)\right\|}$$

where the norm is induced by the scalar product. Considering

$$F_{x_{n-1}x_{n-2}}g_{n-1} = \left\langle \frac{Q\left(x_{n-1}, x_{n-2}, \Gamma\right)}{\left\|Q(x_{n-1}, x_{n-2}, \Gamma)\right\|}, Q(x_{n-1}, x_{n-2}, \Gamma) \right\rangle$$

$$= \left\|Q(x_{n-1}, x_{n-2}, \Gamma)\right\| = \left\|F_{x_{n-1}x_{n-2}}\right\|$$

we can construct the iterative method

$$x_n = x_{n-1} - \frac{F(x_{n-1})}{F_{x_{n-1}x_{n-2}}g_{n-1}} g_{n-1}$$

or

(1.7) $$x_n = x_{n-1} - \left(\frac{\|F(x_{n-1})\|}{\|Q(x_{n-1}, x_{n-2}, \Gamma)\|}\right)^2 Q(x_{n-1}, x_{n-2}, \Gamma).$$

Let F be monotone:

(C) $$F_{x'x''} \geq O_1 \qquad \forall x', x'' \in X$$

and convex

(D) $$F_{x'x''x'''} \geq O_2 \qquad (F_{x'x''} \geq F_{x'x'''} \geq O_1)$$

where O_1 is null-additive (in particular linear), and O_2 is biadditive (in particular bilinear); null-comparable. So, considering the iteration (5.2) and the equality

(1.8) $$F(x_{n-2}) + F_{x_{n-1}x_{n-2}}(x_{n-1} - x_{n-2}) = 0 \qquad x_0 \leq x_{-1} \leq x_{-2}$$

it follows

$$x_n - x_{n-1} =$$

$$= \frac{g_{n-1}}{F_{x_{n-1}x_{n-2}}g_{n-1}}\left[F(x_{n-1}) - F(x_{n-2}) - F_{x_{n-1}x_{n-2}}(x_{n-1} - x_{n-2})\right] =$$

$$= \frac{g_{n-1}}{F_{x_{n-1}x_{n-2}}g_{n-1}} F_{x_{n-1}x_{n-2}x_{n-3}}(x_{n-1} - x_{n-3})(x_{n-1} - x_{n-2}) \leq 0.$$

From this we get

$$x_n - x_{n-1} \leq 0 \qquad (n = 1, 2, \ldots)$$

which means, that the sequence $\{x_n\}$ is monotone decreasing. On the basis (D) and considering (C), it can be shown that the sequence

$$\left\{ F(x_n) \right\}$$

is monotone decreasing, i.e.

$$F(x_n) \le F(x_{n-1}), \quad and \quad 0 \le F(x_n) \quad (n = 1, 2, \dots).$$

If the sequence $\{x_n\}$ bounded from below, then there exists the limit

$$\lim_{n \to \infty} x_n = x^*$$

and considering the continuity of the functional, we get

$$F(x^*) = 0.$$

Remark. The construction of the difference quotient $F_{x_n}(\dots)$ appearing in the iterative method (1.3) is not a problem in the case, when there exists the Fréchet derivative $P'(x)$ then by the notations $P' = \dots$ i.e.

$$F(x) := < P(x), P(x) >, \qquad Q(x_n) := \overline{P'}(x_n) P(x_n)$$

and considering that $\overline{P'}(x_n)$ denotes the adjoint of $P'(x_n)$ we get

$$F'(x_n)\Delta x = < 2Q(x_n), \Delta x >, \qquad F''(x_n)\Delta x^2 = < Q'(x_n)\Delta x, \Delta x >.$$

So, if the approximate element y_n is chosen as follows

$$y_n := \frac{Q(x_n)}{\|Q(x_n)\|}$$

then this yields

$$F'(x_n)y_n = 2 < Q(x_n), \frac{Q(x_n)}{\|Q(x_n)\|} > = 2\|Q(x_n)\| = \|F'(x_n)\|.$$

Thus, the iterative method

(1.9) $$x_n = x_{n-1} - \frac{F(x_{n-1})}{F'(x_{n-1})y_{n-1}} y_{n-1} = x_{n-1} - \frac{F(x_{n-1})}{\|F'(x_n)\| \|y_{n-1}\|} y_{n-1}$$

can be written in the form

(1.9') $$x_n = x_{n-1} - \frac{1}{2}\left(\frac{\|P(x_{n-1})\|}{\|Q(x_{n-1})\|} \right)^2 Q(x_{n-1}).$$

The proof of the convergence of (1.9') can be performed as in the case of (1.3). Starting from (1.9) we obtain a new variant

$$x_{n+1} = x_n - \frac{1}{2}\left(\frac{F(x_n)}{\|F'(x_n)\|} \right)$$

2. Let be

$$f : R^m \longrightarrow R, \quad g : R^m \longrightarrow R^r$$

usually nonlinear functions. Consider the following minimum problem

$$(E) \quad \begin{cases} \min f(x) \\ g(x) \leq \Theta \end{cases}$$

with side condition. Let us introduce the next auxiliary variables

$$u^2 := (u_1^2, \ldots, u_i^2, \ldots, u_r^2)$$

which can be considered as parameters. Then, the problem (E) is equivalent to the next

$$\begin{cases} \min f(\overline{x}) \\ g(x) + u^2 = \Theta \end{cases}$$

where

$$\overline{x} := (x_1, \ldots, x_m, u_1^2, \ldots, u_r^2) \qquad x := (x_1, \ldots, x_m, 0, \ldots, 0)$$

furthermore

$$\overline{f} : R^{m+r} \longrightarrow R^r \qquad F(\overline{x}) := \Big(\overline{f}(\overline{x}), g(x)\Big)$$

and

$$f := (f_1, \ldots, f_i, \ldots, f_k); \qquad g := (g_1, \ldots, g_r).$$

Now let us construct the functional $\overline{F} : R^{m+r} \longrightarrow R$ as follows

$$\overline{F}(\overline{x}) := < F(\overline{x}), F(\overline{x}) > = \sum_{i=1}^{r} f_i^2(x) + \sum_{j=1}^{r} g_j^2(x) + \sum_{k=1}^{r} u_k^4.$$

Therefore, the method (5.2) is applicable. Here, clearly, the approximations $\{x_i\}$ depend on the parameters. If the mapping F or more exactly f and g are differentiable, then instead of (5.2) we may use method (5.6). Illustrative examples: for the differentiability case let us consider the following system of inequalities

$$\min \Big[f(x)\Big] := \min \Big[(x_1 - 1)^2 + (x_2 - 1)^2 - 1\Big]$$

$$g_1(x) := x_1^2 + (x_2 - 1)^2 - 1 \leq 0$$

$$g_2(x) := x_1^2 + x_2^2 - 1 \leq 0$$

introducing the auxiliary variables u_1, u_2 we can transform that problem in following system of equations

$$\min \Big[f(\overline{x})\Big] := \min \Big[(x_1 - 1)^2 + (x_2 - 1)^2 - 1\Big]$$

$$g_1(x; u_1) := x_1^2 + (x_2 - 1)^2 - 1 + u_1^2 = 0$$

$$g_2(\vec{x}, w_2) := x_1^2 + x_2^2 - 1 + w_2^2 = 0$$

we are going to change the functional F in the following way

$$F(\vec{z}) := \left(f(\vec{x}) + \lambda_1 g_1(\vec{x}) + \lambda_2 g_2(\vec{x}), e^{w_1 w_2} \right)$$

where

$$\vec{z} = (x_1, x_2, \lambda_1, \lambda_2, w_1, w_2) \quad and \quad \vec{x} := (x_1, x_2, w_1, w_2).$$

We construct the partial derived differences

$$\frac{\partial F}{\partial x_1} = \frac{\partial f}{\partial x_1} + \lambda_1 \frac{\partial g_1}{\partial x_1} + \lambda_2 \frac{\partial g_2}{\partial x_1} = 2(x_1 - 1) + 2\lambda_1 x_1 + 2\lambda_2 x_1 = 0$$

$$\frac{\partial F}{\partial x_2} = \frac{\partial f}{\partial x_2} + \lambda_1 \frac{\partial g_1}{\partial x_2} + \lambda_2 \frac{\partial g_2}{\partial x_2} = 2(x_2 - 1) + 2\lambda_1 (x_2 - 1) + 2\lambda_2 x_2 = 0$$

$$x_1 = \frac{1}{1 + \lambda_1 + \lambda_2} \qquad x_2 = \frac{1 + \lambda_1}{1 + \lambda_1 + \lambda_2} = (1 + \lambda_1) x_1$$

$$\frac{\partial F}{\partial \lambda_1} = g_1(\vec{x}) = 0; \quad \frac{\partial F}{\partial \lambda_2} = g_2(\vec{x}) = 0$$

i.e.

$$x_1^2 + (x_2 - 1)^2 - 1 + w_1^2 = 0$$

$$x_1^2 + x_2^2 - 1 + w_2^2 = 0$$

$$x_2 = \frac{1}{2} w_2^2 - w_1^2 - 1$$

$$x_2 = \frac{1}{2} w_1^2 - w_2^2 - 1$$

and

$$x_1^2 = 1 - w_2^2 - x_2^2 = 1 - w_2^2 - \frac{1}{4}(w_2^2 - w_1^2 + 1)^2$$

$$\frac{\partial F}{\partial w_1} = \lambda_1 2 w_1 + w_2 e^{w_1 w_2} = 0$$

$$\frac{\partial F}{\partial w_2} = \lambda_2 2 w_2 + w_1 e^{w_1 w_2} = 0$$

$$\lambda_1 = \lambda_2 \left(\frac{w_1}{w_2} \right)^2$$

It may be seen easily the solution set is a circular arc centred at (1,1) and with endpoints at $(\frac{1}{2}, -\frac{1}{2}\sqrt{3})$ *and* (0,1).

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2.§. Proportional distribution of a given amount of money among companies considering the efficiency

Let us now consider this single financial problem. For that purpose let x_j be the portion of money given to the jth company and let $r_j(x_j)$ the efficiency function of this company. So we can formulate this nondifferentiable problem in the following way:

$$\sum_{i=1}^{N} |x_i - a_i| = S \tag{2.1}$$

moreover

$$\frac{|x_1 - a_1|}{r_1(x_1)} = \frac{|x_2 - a_2|}{r_2(x_2)} = \dots = \frac{|x_j - a_j|}{r_j(x_j)} = \dots = \frac{|x_N - a_N|}{r_N(x_N)} = \frac{S}{\sum_{j=1}^{N} r_j(x_j)}. \tag{2.2}$$

where a_j represent the credit of the jth company. The amount of the money for the jth company is

$$|x_j - a_j| = \frac{S r_j(x_j)}{\sum_{i=1}^{N} r_i(x_i)}, \qquad (j = 1, 2, \dots, N)$$

where $r_j(x)$ is the efficiency (profit) for company j.

$$r_j(x) := v_j[1 - (1 - e^{-\alpha_j/x})^x].$$

Here v_j means the maximal possible profit of the jth company and α_j characterises the market competition.

If we want to spend some y money for the advertisement beside the x investment, than

$$r_j(x, y) := v_j[1 - (1 - e^{-\alpha_j/(x+y)})^x] \qquad (j = 1, 2, \dots, N)$$

For the next we denote by

$$M_j := \max r_j'(x) \qquad (j = 1, 2, \dots N)$$

At last we are going to calculate min S. We say min $S = \bar{S}$. Really, using (2.1), (2.2) obtain the differential quotient of S, i.e.

$$S_{x\bar{x}} z = (1 \dots, 1 \dots \dots, 1) z$$

where

$$x := (x_1, x_2, \dots, x_N) \qquad and \quad \bar{x} := (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_N)$$

so we have

$$S_{x\bar{x}} z = |x_1 - a_1| + |x_2 - a_2| + \dots + |x_N - a_N| = S$$

where $z := \Big(|x_1 - a_1|, |x_2 - a_2|, \dots, |x_N - a_N|\Big)$

We mention at this step the absolute function of $|x_i - a_i|(1 = 1, 2,)$ hasn't the derivative so in this case we consider the supporting line for the steepest gradient which are equal to 1.

We may change S also as follows.

$$\sum_{i=1}^{N}(x_i - a_i) = S$$

either

$$\sum_{i=1}^{N}(x_i - a_i)^2 = S$$

or

$$\sum_{i=1}^{N}|x_i| = S$$

and other suitable simple functions

Studding the following relations

$$O < F_{x_n x_{n-1} x_{n-2}}(x_n - x_{n-2})(x_n - x_{n-1})$$

$$= \Big[F_{x_n x_{n-1}} - F_{x_n x_{n-2}}\Big](x_n - x_{n-1}) < F_{x_n x_{n-1}}(x_n - x_{n-1})$$

so

$$F(x_n) = F(x_{n-1}) + F_{x_{n-1} x_{n-2}}(x_n - x_{n-1}) +$$

$$+ F_{x_n x_{n-1} x_{n-2}}(x_{n-1} - x_{n-2})(x_n - x_{n-1}),$$

at last we established a less restrictive formula that of (5.1), i.e.

$$F(x_n) = F(x_{n-1}) + F_{x_{n-1} x_{n-2}}(x_n - x_{n-1}) + F_{x_n x_{n-2}}(x_n - x_{n-1})$$

so we can conclude that the minimum of S exists.

§.3. Nonlinear singular first-order system of differential equations with bilocal conditions

1). Consider the next nonlinear system of differential equations, consisting of m equations

(3.1) $$P(x) := x'(t) - \mathcal{F}\Big(t, x(t)\Big) = \Theta$$

with the bilocal conditions

(3.2) $$x_i(t) = \beta_i, \quad i = 1, 2, \ldots, p \qquad (for \quad t = 0)$$

furthermore

(3.3) $$\psi_j\Big(x(1)\Big) = 0 \qquad j = p+1, p+2, \dots, m \qquad (for \quad t = 1).$$

Here we apply the next notations as usual:

$$x(t) := \Big(x_1(t), \dots, x_m(t)\Big)$$
$$x^0(t) := \Big(x^0_1(t), \dots, x^0_m(t)\Big)$$
$$x^{\cdot}(t) := \Big(x^{\cdot}_1(t), \dots, x^{\cdot}_m(t)\Big)$$
$$\mathcal{F}\Big(t, x(t)\Big) := \left(\tfrac{1}{x_1(t)} f_1\Big(t, x_1(t)\Big), \dots, \tfrac{1}{x_m(t)} f_m\Big(t, x(t)\Big)\right).$$

From the operator equation $P(x) = \Theta$ we construct the functional

$$F(x) := < P(x), P(x) >$$

with the help of scalar product. Then, let us take the first-order difference quotient of F on the pair of knots x_{n-1}, x_{n-2} :

$$\overset{\Delta x}{F_{x_{n-1}x_{n-2}} \cdots} := < P(x), P(x) >_{x_{n-1}x_{n-2}} \overset{\Delta x}{\cdots} =$$

$$= < \overset{\Delta x}{P_{x_{n-1}x_{n-2}}} \dots P(x_{n-2}) > +$$

$$+ < \overset{\Delta x}{P_{x_{n-1}x_{n-2}}} \dots, \Big(P(x_{n-1}) + P(x_{n-2})\Big) > =$$

$$= < \overset{\Delta x}{\dots}, P^T_{x_{n-1}x_{n-2}}\Big(P(x_{n-1}) + P(x_{n-2})\Big) > .$$

Furthermore, let be

$$Q(x_{n-1,n-2}) := P^T_{x_{n-1}x_{n-2}}\Big(P(x_{n-1}) + P(x_{n-2})\Big)$$

and

(o) $$y_{n-1} := \frac{Q(x_{n-1}, x_{n-2})}{\Big\| > Q(x_{n-1}, x_{n-2}) > \Big\|}.$$

Let us construct the following iterative method:

(1) $$x_n = x_{n-1} - \frac{F(x_{n-1})}{F_{x_{n-1}x_{n-2}}(y_{n-1})} y_{n-1}.$$

Applying the difference quotient $F_{x_{n-1}n-2}$ we get

(2) $$F(x_{n-1}) + F_{x_{n-1}x_{n-2}}(x_n - x_{n-1}) = 0$$

therefore, the equation (1) gives

$$x_n - x_{n-1} = -\frac{y_{n-1}}{F_{x_{n-1}x_{n-2}}(y_{n-1})}\Big[F(x_{n-1}) - F(x_{n-2}) -$$

$$-F_{x_{n-2}x_{n-3}}(x_{n-1} - x_{n-2})\Big] =$$

$$= -\frac{y_{n-1}}{F_{x_{n-1}x_{n}y_{n-1}}} F_{x_{n-1}x_{n-2}x_{n-3}}(x_{n-1} - x_{n-2})(x_{n-1} - x_{n-2}).$$

Assuming that the difference quotient of F is positive, follows that $x_n - x_{n-1} < \Theta$. Let us consider that

$$\ldots - F_{x_{n-1}x_{n-2}} \frac{\Delta x}{\ldots} \frac{\Delta x}{\ldots} - F(x_{n-1}) = \Theta$$

where $\Delta x_i(\Theta) = \Theta$, furthermore, that

$$\psi_{x_{n-1}x_{n-2}}(x_n - x_{n-1}) + \psi(x_{n-1}) = 0$$

If $\mathcal{F}$ and ψ are monotone and convex in the general sense we may construct the so-called index of performance as follows

$$F = \int_0^1 \left\|x' - \mathcal{F}\Big(t, x(t)\Big)\right\|^2 dt + \left\|\psi\Big(x(t)\Big)\right\|^2$$

It can be seen that there exist a minimum of F, which means that there exist a solution of our bilocal problem. Remarks

1. There exists solution for the m-order system too, because this system can be transformed into a first-order system as we have already seen.

2. There exists a solution also in the case of the system of differential equations with delayed arguments.

3. The problems (1)-(3) may depend on the parameter, if F and $\Phi$are continuous and Lipschitzfunctions, and if we can construct a pseudometric. In this case our methodology can be applied and it guarantees the existence of the solution. The papameter belongs to the set of real numbers but it can be an element of an other set too, as we shall wee later.

2). Illustrative numerical example: For this purpose we consider a very simple nonlinear systems.

$$x_1' = 10x_2; \quad x_2' = 10x_3; \quad x_3' = 5x_2x_1$$

with the following bilocal conditions

$$x_1(O) = O, \quad x_2(O), \quad x_3(O) = ?$$

$$x_1(1) = ? \quad x_2(1) = 1, \quad x_3(\wedge) = ?$$

$$x = (x_1, x_2, x_3)^T, \quad x' := (x_1', x_2', x_3')$$

we construct the functional $F(x)$ as follows

$$F(x) := \int_0^1 \Big[(x_1' - 10x_2)^2 + (x_2' - 10x_3)^2 + (x_3' - 5x_2x_3)\Big]dt + (x_2 - 1)^2$$

then

$$F'(x)\ldots = \int_0^1 \Big[2x_1' \ldots, (2x_2' - 10x_3) \ldots - (20x_2 - x_2' - 10x_2)\ldots\Big]dt +$$

$$\int_0^1 \Big[\ldots, 2x_2 \ldots\ldots\Big]dt$$

Using the formulas (α) (1), (2), and (α) we get

(α)
$$y_{n-1} := \frac{Q(x_{n-1}, x_{n-2})}{\|Q(x_{n-1}, x_{n-2})\|}$$

(1)
$$x_n = x_{n-1} - \frac{F(x_{n-1})}{F_{x_{n-1}x_{n-2}}(y_{n-1})} y_{n-1}.$$

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§.4. A Physical Numerical Illustration[1]

Wording. A study of the controlled fusion of plasmas relies either on discrete models such as the Vlasov equations, or on continuous models derived from the magneto-hydrodynamic, denoted by M.H.D. equations. In the latter case the numerical simulation of the plasma corresponds to the solution of the 3-dimensional M.H.D. system with one (or perhapse two) fluid(s) and is out of reach at this time. However the specialized literature contains a long list of accessible some specific phenomena.[1].

In the next we try to recall the equations of the equilibrium which lead to a free boundary value problem which may possess several solutions.

The Free Boundary Value Problem of Plasma Physics. We recall the formulation of the problem given by C. Mercier [11] (cf. also the Appendix of [1]).

Let $0z$ be the axis of the machine. In a cross-section plan $0rz$, we call ω the cross-section of the machine; its boundary Γ is the cross-section of the shell. The plasma fills a part Ω_p of Ω whose boundary is denoted Γ_p and the complementary region $\Omega_v = \Omega - (\Omega_p \cup \Gamma_p)$ is emty.

We shall follow [1]. Starting the fact that in Ω_v we have the Maxwell equations

$$div\ B = 0, \qquad curl\ B = 0. \tag{4.1}$$

In cylindric coordinates r, θ, z the first equation (4.1) implies the existence of a function u such that

$$B_r := \frac{1}{r}\frac{\partial u}{\partial z}, \qquad B_z := -\frac{1}{r}\frac{\partial u}{\partial r} \tag{4.2}$$

and the second equation (4.1) implies

$$\mathcal{L}u := \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 u}{\partial z^2} = 0 \tag{4.3}$$

In the plasma Ω_p, the M.H.D. equations at equilibrium reduce to

$$div\ B = 0, \qquad curl\ B = \mu_o J, \qquad grad\ p = J \times B. \tag{4.4}$$

The first equations (4.4.) shows the existence of a functions u satisfying again (4.2). Some simple calculations using the axialsymmetry imply that

$$grad\ p \parallel grad\ u, \ \text{ i.e. } \ p = p(u), \tag{4.5}$$

and on the other hand we have the equation (I.)

$$\overline{\mathcal{L}}u := \mathcal{L}u + \mu_o \frac{dp}{du} = 0 \tag{4.6}$$

which is called the Grad-Shafranov equations.[24,25]

The function $p := p(u)$ is similar to a constitutive equation for the present plasma and must be considered as given. In the simplest physical model, $p = p(u)$ is a quadratic function

$$p(u) := a_o + a_1 u + a_2 u^2 \tag{4.7}$$

and then $\frac{dp}{du} = 2a_2 u$, the coefficient a_1 vanishing because of physical assumptions. Some more complicated functions $p(u)$ will be considered in Section.

The boundary conditions are (II.)

$$B.\nu = 0 \ \text{on} \ \Gamma_p \ \text{and} \ \Gamma, \ B.\tau \quad \text{continuous across} \ \Gamma_p \tag{4.8}$$

It follows that u is constant on Γ_p and Γ (we choose $u = 0$ on Γ_p) and $\frac{\partial u}{\partial \nu}$ is continuous across Γ_p. Furthermore $u \neq 0$ does not vanish in Ω_p (physical assumption) and

$$\int_{\Gamma_p} \frac{1}{r}\frac{\partial u}{\partial \nu} dl = I > 0 \tag{4.9}$$

is given.

Observation. In order simplify somehow the notations we replace $\mathcal{L}$ by Δ and we modify accordingly the equations (the tours is replaced by an infinite cylinder). We set also $\lambda = 2\mu_o a_2$. This leads us to the following free boundary value problem (where Γ_p is considered unknown) :

$$\Delta u = 0 \ \text{in} \ \Omega_v \tag{4.10}$$

$$\overline{\mathcal{L}}(u) := \Delta u + \lambda u = 0 \ \text{in} \ \Omega_p \tag{4.11}$$

$$u = 0 \quad \text{on} \quad \Gamma_p \tag{4.12}$$

$$\frac{\partial u}{\partial \nu} \tag{4.13}$$

continous across on Γ_p

$$u = \text{unknown constant on } \Gamma \tag{4.14}$$

$$\int_{\Gamma_p\cap\Gamma} \frac{\partial u}{\partial \nu} d\,l = I > 0 \tag{4.15}$$

$$u \neq 0 \ in\ \Omega_p. \tag{4.16}$$

By the maximum principle, we have:

$$\begin{cases} \Omega_p = \{x, u(x) < 0\} \\ \Omega_e = \{x, u(x) > 0\} \\ \Gamma_p = \{x, u(x) = 0\} \end{cases} \tag{4.17}$$

so that the free boundary is known once function u is known. It follows also from (4.16) and (4.17) that λ is the first eigenvalue of the Dirichlet problem in Ω_p.

Remark. Here we asume that the contour of the function does not touch the contour Γ.

If there exists the Fréchet derivative $\overline{P}'(x)$ and its adjoint $\overline{P}'^T$ then we may introduce the functional F, i.e.

$$F(x) := <\overline{P}(x), \overline{P}(x)> = O, \qquad Q(x_n) := \overline{P}'^T(x_n)\overline{P}(x_n).$$

Therefore

$$F'(x_n)x = <2Q(x_n), \Delta x>, \qquad F''(x_n)\Delta x^2 = <Q'(x_n)\Delta x, \Delta x>.$$

If the approximate element is changed in suitable manner i.e.

$$y_n := \frac{Q(x_n)}{\left\|Q(x_n)\right\|}$$

so we have

$$F'(x_n)y_n = 2 < Q(x_n), \frac{Q(x_n)}{\left\|Q(x_n)\right\|} >= 2\left\|Q(x_n)\right\| = \left\|F'(x_n)\right\|.$$

and

$$x_n = x_{n-1} - \frac{F(x_{n-1})}{F'(x_{n-1})y_{n-1}} y_{n-1} (5.6)$$

more concretely

$$x_n = x_{n-1} - \frac{1}{2}\left(\frac{\|P(x_{n-1})\|}{\|Q(x_{n-1})\|}\right)^2 Q(x_{n-1}) (5.2')$$

The convergence may be proved analogously. i.c.

$$x_n - x_{n-1} =$$

$$= -\frac{y_{n-1}}{F'(x_{n-1})y_{n-1}}\left[F(x_{n-1} - F(x_{n-2}) - F'(x_{n-2})(x_{n-1} - x_{n-2})\right] \leq$$

$$\leq -\frac{y_{n-1}}{2F'(x_{n-1})y_{n-1}} F''(\xi)(x_{n-1} - x_{n-2})^2.$$

So and

$$x_n - x_{n-1} \leq 0 \qquad (n = 1, 2, \dots)$$

what means that x_n is a monotone decreasig sequence.

It can be seen easily that the sequence

$$\left\{F(x_n)\right\}$$

also monoton decreasing, i.e.

$$F(x_n) \leq F(x_{n-1}), \quad and \quad 0 \leq F(x_n) \qquad (n = 1, 2, \dots).$$

In that situation the sequence $\{x_n\}$ is bounded from below and exists

$$\lim_{n\to\infty} x_n = x^*$$

considering the monotonity of the functional $F(x)$, it follows

$$F(x^*) = 0.$$

In that case we may consider as functional F in following way

$$F(v) := < P(v), P(v) > = 0$$

where

$$P(v) := \lambda \int_\Omega (\nabla v)^2 dx + \lambda \int_\Omega (v_t)^2 dx - Iv(\Gamma) + V_{max}$$

Optimality of a given parallelepiped.

1. Being given the parallelepiped having the volume $V(x_1,x_2,x_3) :=:= x_1x_2x_3 = a^3$ with his dimentions x_1, x_2, x_3 we will determine the minimal surface $S(x_1,x_2,x_3) := 2x_1x_2 + 2x_1x_3 + 2x_2x_3$. So we construct the function

$$F(x_1,x_2,x_3) := 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + \lambda(x_1x_2x_3 - a^3)$$

appliing the classical Lagrange multiplier λ we get

$$\frac{\partial F}{\partial x_1} = 2x_2 + 2x_3 + \lambda x_2x_3 = 0, \qquad \frac{\partial F}{\partial x_2} = 2x_1 + 2x_3 + \lambda x_1x_3 = 0$$
$$\frac{\partial F}{\partial x_3} = 2x_1 + 2x_2 + \lambda x_1x_2 = 0, \qquad \frac{\partial F}{\partial \lambda} = x_1x_2x_3 - a^3 = 0.$$

So we can get the extremal point

$$x_1 = x_2 = \frac{2a}{\sqrt[3]{4}}, x_3 = \frac{a}{\sqrt[3]{4}},$$

wherer the multiplier

$$\lambda = -\frac{2\sqrt[3]{4}}{a}$$

because the convexity of F the surface $S(x_1,x_2,x_3)$ has a minimum for

$$\overline{x}_1 = \frac{2a}{\sqrt[3]{4}}, \quad \overline{x}_2 = \frac{2a}{\sqrt[3]{4}}, \quad \overline{x}_3 = \frac{2a}{\sqrt[3]{4}}.$$

For the maximal volume of above parallelepiped we consider the functional F

$$F(x_1,x_2,x_3,\lambda) = x_1x_2x_3 + \lambda(x_1 + x_2 + x_3 - a)$$

and apliing in that case the Lagrangean multiplier we can obtain the extremal point

$$\frac{\partial F}{\partial x_1} = x_2x_3 + \lambda = 0, \qquad \frac{\partial F}{\partial x_2} = x_1x_3 + \lambda = 0,$$
$$\frac{\partial F}{\partial x_3} = x_1x_2 + \lambda = 0, \qquad \frac{\partial F}{\partial \lambda} = x_1 + x_2 + x_3 - a = 0$$

where the extremal point $x_1 = x_2 = x_3 = \frac{a}{3}$ where of course $\lambda = -\frac{a^2}{9}$. Because the negativity of F the volume V has a maximum for the above point,

$$d^2F \leq -\frac{a}{3}[dx_2^2 + dx_2dx_3 + dx_3^2] < 0$$

that holds

$$V_{max} = \frac{a^3}{27} \quad for \quad \overline{x}_1 = \overline{x}_2 = \overline{x}_3 = \frac{a}{3}$$

Let us observe that kind of parallelepiped intervene in numerical problems plasma physics. Similarly we may pose the following optimality as well

2. If we have to construct the maximum of surface $S(x_1, x_2, x_3)$ i.e.

$$\min S(x_1, x_2, x_3)$$

Satisfiing the condition $x_1 + x_2 + x_3 = a$

we consider the function $F(x_1, x_2, x_3, \lambda) = 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + 2x\lambda(x_1x_2x_3 - a^3)$ we get by Lagrangen multiplier λ

$$\frac{\partial F}{\partial x_1} = 2x_2 + 2x_3 + \lambda_2 x_3 = 0, \qquad \frac{\partial F}{\partial x_2} = 2x_1 + 2x_3 + \lambda x_1 x_3 = 0$$

$$\frac{\partial F}{\partial x_3} = 2x_1 + 2x_2 + \lambda x_1 x_2 = 0 \qquad \frac{\partial F}{\partial \lambda} = x_1x_2x_3 - a^3 = 0$$

having the solving

$$\overline{x}_1 = \overline{x}_2 = \frac{2a}{\sqrt[3]{4}}, \qquad \overline{x}_3 = \frac{a}{\sqrt[3]{4}}, \qquad \lambda = -\frac{2\sqrt[3]{4}}{a}$$

Beeing at last

$$d^2F = 2(dx_1^2 2 + dx_2^2 + dx_1 dx_2) > 0$$

results

$$\overline{x}_1 = \frac{2a}{\sqrt[3]{4}}, \qquad \overline{x}_2 = \frac{2a}{\sqrt[3]{4}} \qquad \overline{x}_3 = \frac{a}{\sqrt[3]{4}}$$

i.e.

$$S(\overline{x}_1, \overline{x}_2, \overline{x}_3)$$

represents a minimum.

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§.5. Inventory Problems

There are a large number of processes which involve ordering supplies at the present, so as to anticipate an unknown demand in the future. We shall treat here only the simple case where the distribution of demand is assumed to known.

Let us consider a process involving the stocking of a m different item. We assume that orders for further supplies are made at each of a finite set of times, and immediately fulfilled. After the order has been made and filled, there is a demand made for the item. This demand is satisfied as far as possible, with any excess of demand over supply leading to a penalty cost. Let us consider the mathematical model

$$x_i(t) = \min_{y>t}[k_i(y-t) + \int_y^\infty p_i(s-y)\varphi_i(s)ds +$$

$$+ x_i(0)\int_y^\infty \varphi_i(s)ds + \int_0^y x_i(y-s)\varphi_i(s)ds] := A_i(t) \tag{5.1}$$

$$(i = 1, 2, ..., m)$$

We suppose the folloing notation are known:

(a) $\varphi_i(s)ds$ = the probability that the demand will lie between s and $s+ds$. (b) $k_i(t)$ = the cost of ordering t items to increase the stock level. (c) $p_i(t)$ = the cost of ordering t items to meet an excess, t, of demand over supply, the penalty cost. (d) $x_i(t)$ = the probability constant for the stock level. To simplify the situation, let us assume that these functions are independent of time. Our aim is to assure the solvability for the system (1). Let us suppose that we order at the first stage a quantity $y-x$ to bring the level up to y. Then the expected cost is given by the function

$$k_i(y-t) + \int_y^\infty p_i(s-y)\varphi(s)ds; \quad (i = 1, 2, ..., n)$$

Hence, we obtain (1).

Remark. The functions $k(t)$, $p(t)$ and $\varphi(s)$, so and consider

$$k(t) := kt, \qquad k > 0,$$

$$p(t) := pt, \qquad p > 0.$$

Let denote $T_i(y, t; x_i)$ the expression in the square bracket of the right hand side of (1). So, we can write

$$x_i(t) = \min_{y \geq t} T_i(y, t; x_i) := A_i(t).$$

In this case the probability cost can be given in the form

$$k_i(y-t) + \int_y^\infty p_i(s-y)\varphi_i(s)ds.$$

We are looking for the solutions $x_i(t)$ of the equation (1) in the class of the continuous and bounded functions on the half line $x \geq 0$ and which class is denoted by $C[0,\infty)$.

Assume that the following conditions are hold:

1^0. $k_i(t)$ is a given monotone and continuous function on $[0,\infty)$ and $k_i(z) \to \infty$, if $z \to \infty$;

2^0. $p_i(t)$ is a given monotone and continuous function on $[0,\infty)$ and

$$\int_0^\infty p_i(s)\varphi_i(s)ds < +\infty; \qquad (i = 1, 2, ..., m)$$

3^0. $\varphi_i(t)$ is a nonnegative integrable function on $[0,\infty)$ and

$$\int_0^\infty \varphi_i(s)ds = 1.$$

By these conditions, there exists a unique solution of (1), which is continuous and finite on the halfline $[0,\infty)$.

For the proof, let us denote the mapping $A(x(t))$ as follows:

$$A(x(t)) := \min_{t \leq y \leq x} T(y, t; f) \qquad (i = 1, 2, ..., m) \tag{5.2}$$

$$x(t) := (x_1(t), ..., x_i(t), ..., x_m(t)); \qquad A(t) := (A_1(t), ..., A_m(t)).$$

We assume that $k_i(y-t) \to +\infty$, if $y \to +\infty$. The other term of the expression $T(y, t; f)$ are remain bounded, so easy to see that there exists a y, where there is an infimum (which depend on the t and the function f). Choosing

$$F(x) := < P(x), P(x) >$$

$$P(x) := x(t) - A\Big(x(t)\Big)$$

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§.6. Optimal control

Consider the following system of differential equations consisting of m equations (Cauchy problem).

$$\frac{dx}{dt} = f(t, x, u), \quad (x(0) = x^0), \tag{5.1}$$

where $x(t)$ and $f(t, x(t), u(t))$ are vectors with components m, i.e. $x, f \subset R_m$ and $u = u(t)$ is a control parameter with components r, i.e. $u(t) \in U$, but $\subset R^r$, i.e. $t \in R$ and $f : R \times R^m \times R^r \longrightarrow R^m$ is continuous and Lipschitzfunction with respect to the vectors $x(t)$ and $u(t)$. Here, clearly, R^m or R^r denote m-dimensional or r-dimensional spaces, and R is the set of real numbers. Finally, the neighbourhood U depend on t and $x(t)$, i.e. $U = U(t, x(t))$ The aim of the optimal control is to determine a solution x^*, if it is a given vector $x^0 := (x_1^0(t), \dots, x_m^0(t))$, and for $u(t) \in U(t), x$, so, that the solution x^* would realize the smallest time. The next we deal with the question of solvability of this control problem. We have already dealt with the system of the differential equations with paramerter. The difference is that now u is a r-dimensional vector not a real number, and that in this case we have a minimum-problem. Let us construct the performance-index as follows

$$F(t, u(t)) = V(t, u(t)) +$$

$$+ \int_0^t \left\| f\left(t, x(t), u(t)\right) \right\| dt$$

Let assume that V and f are monotone and convex in the general sense. It can be seen, as in the paragraph before, that the performance-index there exists a minimum.

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§ 7. The problem of vertebrate retina

1. Introduction. In the present paper we consider the mathematical model formulated by Oguztoreli, for studying the neural activities in a vertebrate retina has been investigated, where the basic network contains five interconnected neurons: a receptor cell, a bipolar cell, a horizontal cell, an amacrine cell, and a retinal ganglion cell. More recently, in (Oguztoreli and O'Mara, 1980), the basic network has been extended to a larger network containing twelve neurons. In both of these works, the performances of the basic and extended models were discussed under different structural and processing conditions with constant inputs by using the results of one of our earlier work (Oguztoreli, 1979). In the present paper we study by simulations the responses of the basic retinal network to piecewise constant and periodic inputs. The step and frequency responses of the extended retinal network will be discussed in a forthcoming paper.

The functional organization of vertebrate retina has been extensively investigated in the literature (cf. Cervetto and Fuortes, 1978; Dowling, 1970, 1972; Dowling and Ehinger, 1978; Dowling and Werblin, 1969; Gerschenfeld and Piccolino, 1980; Hedden and Dowling, 1978; Lasbovic, 1978; Michael, 1969; Perry, 1979; Piccolino and Gerschenfeld, 1980; Shepherd, 1979; Werblin 1976, 1973; Wu and Dowling, 1980). In two recent works (Oguztoreli, 1980; Oguztoreli and O'Mara, 1980) we have investigated two idealized retinal networks based on the findings in the above mentioned literature by implementing the rules and results of our earlier work (Oguztoreli, 1979). The basic retinal network considered in (Oguztoreli, 1980) is described schematically in Fig.1, where

In the next we shall denote by σ_{ij} is the total time delay in the transfer of the action of neuron ◯ on neuron ◯ due to cell time constants, cable spread, and kinetics of release and of transmitter action.

We assume that $c_{ii} = 0$ and $\sigma_{ii} = 0$

Let $x_i(t)$ be the activity function (the normalized firing rate) of neuron ◯ at time t. It was shown in (Oguztoreli, 1980) that the basic network is described mathematically by following systems of functional differential equations:

$$(7.1)\begin{cases} \frac{1}{\alpha_{10}}\frac{dx_1}{dt} + x_1 = S\Big\{ f_1 + c_{12}x_2(t-\sigma_{12}) + c_{13}x_3(t-\sigma_{13}) + \\ + b_{11}\int_0^t e^{-\alpha_{11}(t-\tau)} x_1(\tau)d\tau \Big\} \\ \frac{1}{\alpha_{20}}\frac{dx_2}{dt} + x_2 = S\Big\{ c_{21}x_1(t-\sigma_{21}) + c_{23}x_3(t-\sigma_{23}) + \\ + c_{24}x_4(t-\sigma_{24}) + c_{25}x_5(t-\sigma_{25}) + b_{21}\int_0^t e^{-\alpha_{21}(t-\tau)} x_2(\tau)d\tau \Big\} \\ \frac{1}{\alpha_{30}}\frac{dx_3}{dt} + x_3 = S\Big\{ c_{32}x_2(t-\sigma_{32}) + \\ + c_{35}x_5(t-\sigma_{35}) + b_{31}\int_0^t e^{-\alpha_{31}(t-\tau)} x_3(\tau)d\tau \Big\} \\ \frac{1}{\alpha_{40}}\frac{dx_4}{dt} + x_4 = S\Big\{ c_{42}x_2(t-\sigma_{42}) + b_{41}\int_0^t e^{-\alpha_{41}(t-\tau)} x_4(\tau)d\tau \Big\} \\ \frac{1}{\alpha_{50}}\frac{dx_5}{dt} + x_5 = S\Big\{ c_{52}x_2(t-\sigma_{52}) + \\ + c_{53}x_3(t-\sigma_{53}) + b_{51}\int_0^t e^{-\alpha_{51}(t-\tau)} x_5(\tau)d\tau \Big\}, \end{cases}$$

where

$$(7.2) \qquad S(u) = [1 + e^{-u}]^{-1}$$

and

α_{i0} is the rate constant characterized by the fact that a step change in input to neuron i produces an exponential approach from the initial value $x_i(0)$ to a steady-state firing rate with the rate constant α_{i0};

b_{i1} is an adaption or self-inhibition factor for neuron (i) in the case $b_{i1} < 0$ and is a self-excitation factor in the case $b_{i1} > 0$ with the rate constant $\alpha_{i1}\ (> 0)$.

Let $\phi_i(t), 0 \le \phi_i(t) \le 1, (i = \overline{1,5})$, be certain sufficiently smooth functions defined on the interval

$$(7.3) \qquad I_0 = [-\sigma, 0] = \Big\{ t \mid -\sigma \le t \le 0,\ \sigma = \max_{ij} \sigma_{ij} \Big\}.$$

Then the system (1.1) admits a unique solution $\{x_i(t) \mid i = \overline{1,5}\}$ for $t \ge 0$ such that

$$(7.4) \qquad \begin{cases} x_i(t) = \phi_i(t), & (t \in I_0) \\ 0 \le x_i(t) \le 1 & (t \ge 0) \end{cases}$$

(cf. Oğuztöreli, 1979,1980). As in the article (Oğuztöreli, 1980), we shall consider the following parameter configuration as "standard" for reference and comparison:

$$
\begin{aligned}
&a_{1i} = 100\\
&a_{ii} = 10\\
&\sigma_{ij} = \sigma = 0.005\\
&\Delta_i(t) \equiv 0 \quad (-\sigma \le t \le 0)\\
(7.5) \qquad &b_{15} = -2500,\ b_{21} = -1500,\ b_{31} = -1250\\
&b_{41} = -1000,\ b_{51} = -750\\
&c_{12} = c_{13} = c_{24} = c_{25} = c_{32} = -1\\
&c_{21} = c_{23} = c_{31} = c_{42} = c_{52} = c_{54} = 10\\
&f = 50 \quad (i = \overline{1,5})
\end{aligned}
$$

The basic retinal network with standard parametric configuration will be called the standard basic network and will be abbreviated as SBN.

The fact that the neurons in a simple retina are all of the second-order in the sense Stein et al. (1973, 1974), and some of other functional properties of vertebrate retinal have been briefly outlined in our earlier works (Oğuztöreli, 1980; Oğuztöreli and O'Mara, 1980). In both of these works the performances of the basic and extended retinal models were discussed under different conditions with constant inputs. In the present work we investigate the step and frequency responses of the basic retina model by simulations. The step and frequency response of the extended network will be discussed in a forthcoming paper. Throughout this work we use the terminology, notation, and results of Oğuztöreli(1979,1980). Using the AMDAHL 470 V/7 computing system of the University of Alberta we made a considerable number of simulations. Because of the limited scope of the present paper only a small part of these simulations will be exhibited below. All the graphs are restricted to the time interval $0 \le t \le 1.5$.

The basic network described in Fig.1. can be easily adjusted to the special conditions of real situations by adding and/or deleting some of the connectivity pathways marked by arrows, and/or by determining the processing and network parameters using experimental data (cf. Oğuztöreli, 1980).

Remark. Let us mention that we solved a system compound by a number of 1100 differential equations of the same type.

Remark 2. The given system (5) may be noted $P(x) = 0$. So for the solving of $P(x)$ construct the functional

$$\max_{x \in K} \langle P(x), P(x) \rangle = 0$$

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§. 8. Large nonlinear dynamical systems. Global economic problems

Consider the following systems

$$\tilde{P}_j^{(i)}(X) := \frac{dX(t)_j^{(i)}}{dt} - f_j^{(i)}\Big(X(t)_1^{(i)}, \ldots, X(t)_j^{(i)}, \ldots, X(t)_{n_i}^{(i)}; S^{(i)}(t)\Big) = 0$$

$$1 \leq j < n_i$$

where $S^i(t)$ represents economic structure and the sets $X_j^{(i)}, 1 \leq j \leq n_i$ are the state variable; and $\mathcal{X} \ni X := (X_1 \ldots, X_{n_i})$; moreover consider the following "restrictions"

$$\overline{P}_k^{(i)}(X) := g_k^{(i)}\Big(X(s)_1^{(i)}(s), \ldots, X(s)_j^{(i)}, \ldots, X(s)_{n_i}^{(i)}(s); S_{(s)}^{(i)}\Big) = 0,$$

$$s \in [t, t+\theta_i], 1 \leq k \leq n_i$$

Choosing at last the suitable manner the functional

$$F(X) := \sum_j \Big[\tilde{P}_j^{(i)}(X)\Big]^2 + \sum_k \Big[\overline{P}_k(X)\Big]^2$$

we can construct the iteration method

$$x_n = x_{n-1} - \frac{F(x_{n-1})}{F'(x_{n-1})y_{n-1}} y_{n-1}$$

or

$$X_n = X_{n-1} - \frac{1}{2}\left(\frac{P(x_{n-1})}{\|P'(x_{n-1})\|}\right)$$

where

$$P(X) := \sum_i \tilde{P}(X) + \sum_k \overline{P} \subset X$$

is denoted.

We remark that can be seen easily that in the conditions of convexity there exists a functional Φ so that

$$F_x' \leq \Phi$$

for any $x \in X$.

In that situation the convergence of our iterations method (γ) is assured and the posed problem (α) and (β) has a solution.

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