

Dedicated to Professor Iulian Coroian on his 60th anniversary

On the derivative-interpolating spline functions of even degree

COSTICĂ MUSTĂŢA

The aim of the present note is to show that the derivative-interpolating spline functions, considered in some recent papers ([2],[6]) are in fact primitives, chosen in an appropriate way, of interpolating natural spline functions ([3]).

This fact allows to derive some properties of derivative-interpolating spline functions of even order from the corresponding properties of interpolating natural spline functions.

Let $m, n \in \mathbb{N}$, $m \leq n$, and $[a, b]$ an interval contained in $\mathbb{R}$. Let also

$$(1) \quad \Delta_n := a < x_1 < x_2 < \dots < x_n < b$$

be a fixed partition of the interval $[a, b]$.

Definition 1 A function $s : [a, b] \rightarrow \mathbb{R}$ verifying the conditions

$$1^0 \ s \in C^{2m-2}[a, b],$$

$$2^0 \ s \in \mathcal{P}_{m-1} \text{ on } (a, x_1) \text{ and } (x_n, b),$$

$$3^0 \ s \in \mathcal{P}_{m-1} \text{ on } (x_i, x_{i+1}), \ i = 1, 2, \dots, n-1,$$

where $\mathcal{P}_k$ stands for the set of polynomials of degree at most k ($k \in \mathbb{N}$), is called an interpolating natural spline function (associated to the partition Δ_n).

Denoting by $S_{2m-1}(\Delta_n)$ the set of all interpolating natural spline functions, one sees that $S_{2m-1}(\Delta_n)$ is an n -dimensional subspace of the linear space $C^{2m-2}[a, b]$.

If $s \in S_{2m-1}(\Delta_n)$ then s is of the form

$$(2) \quad s(x) = \sum_{i=0}^{m-1} B_i x^i + \sum_{i=1}^n b_i (x-x_i)_+^{2m-1},$$

where

$$(3) \quad \sum_{k=1}^n b_k x_k^j = 0, \quad j = 0, 1, \dots, m-1$$

(see [3]).

If $Y = (y_1, y_2, \dots, y_n)$ is a fixed vector in $\mathbb{R}^n$ then there exists exactly one function $s_Y \in S_{2m-1}(\Delta_n)$ verifying the equalities:

$$(4) \quad s_Y(x_i) = y_i, \quad i = 1, 2, \dots, n$$

see ([3]).

Let

$$(5) \quad H_{2,Y}^m[a, b] := \left\{ f \in C^{m-1}[a, b] : f^{(m-1)} \text{ is absolutely continuous and } f^{(m)} \in L_2[a, b] \right\}$$

and

$$(6) \quad H_{2,Y}^m[a, b] := \{ f \in H_{2,Y}^m[a, b] : f(x_i) = y_i, \quad i = 1, 2, \dots, n \}.$$

Then

$$(7) \quad H_{2,Y}^m[a, b] \cap S_{2m-1}(\Delta_n) = \{s_Y\}$$

and the functional $J : H_{2,Y}^m[a, b] \rightarrow \mathbb{R}_+$ given by

$$(8) \quad J(f) = \int_a^b \left[f^{(m)}(x) \right]^2 dx, \quad f \in H_{2,Y}^m[a, b]$$

has the property

$$(9) \quad \min \{ J(f) : f \in H_{2,Y}^m[a, b] \} = J(s_Y)$$

i.e. the minimum of the L_2 -norms of the derivatives of order m of the functions in $H_{2,Y}^m[a, b]$ is attained at the interpolating natural spline function s_Y ("the minimal norm property").

Also, for each $f \in H_{2,Y}^m[a, b]$ the inequality

$$(10) \quad \|f^{(m)} - s_Y^{(m)}\|_2 \leq \|f^{(m)} - s\|_2$$

holds for any $s \in S_{2m-1}(\Delta_n)$ ("the best approximation property") (see [3]).

Now, we shall introduce the derivative-interpolating spline functions of even order $2m$, having properties similar to (9) and (10). (see [2]).

Let $m, n \in \mathbb{N}$, $m \leq n+1$, and Δ_n the partition (1) of the interval $[a, b]$.

Definition 2 ([2]). A function $S : [a, b] \rightarrow \mathbb{R}$ is called a *natural spline function of order $2m$* if it verifies the conditions:

$$I^0 S \in C^{2m-1}[a, b],$$

$$I^0 S \in \mathcal{P}_m \text{ on } (a, x_1) \text{ and } (x_n, b),$$

$$I^0 S \in \mathcal{P}_{2m} \text{ on } (x_i, x_{i+1}), i = 1, 2, \dots, n-1.$$

The set of all interpolating natural spline functions of order $2m$ will be denoted by $\mathcal{S}_{2m}(\Delta_n)$. It follows that $\mathcal{S}_{2m}(\Delta_n)$ is an $(n+1)$ -dimensional subspace of $C^{2m-1}[a, b]$ (see [6]).

If $Y = (y_0, y_1, \dots, y_n)$ is a fixed vector in $\mathbb{R}^{n+1}$ then there exists only one function $S_Y \in \mathcal{S}_{2m}(\Delta_n)$ verifying the conditions

$$(11) \quad S_Y(\alpha) = y_0$$

$$S_Y(x_i) = y_i, \quad i = 1, 2, \dots, n,$$

where $x_i, i = \overline{1, n}$ are the nodes of the partition Δ_n given by (7).

The function $S_Y \in \mathcal{S}_{2m}(\Delta_n)$ verifying the condition (7) is called the *derivative-interpolating spline of even order $2m$ associated to the vector Y* and to the partition Δ_n .

Any function $S \in \mathcal{S}_{2m}(\Delta_n)$ admits the representation

$$(12) \quad S(x) = \sum_{k=0}^m A_k x^k + \sum_{k=1}^n \alpha_k (x - x_k)_+^{2m-k},$$

$$(13) \quad \sum_{k=0}^n \alpha_k x_k^j = 0, \quad j = 0, 1, \dots, m-1$$

see ([2] or [6]).

Let

$$(14) \quad H_2^{m+1}[a, b] := \left\{ f \in C^m[a, b], f^{(m)} \text{ is absolutely continuous} \right. \\ \left. \text{and } f^{(m+1)} \in L_2[a, b] \right\}$$

and

$$(15) \quad H_{2Y}^{m+1} := \left\{ g \in H_2^{m+1}[a, b] : g(\alpha) = y_0 \text{ and} \right. \\ \left. g'(x_i) = y_i, i = 1, 2, \dots, n \right\}.$$

Then (see [2])

$$(16) \quad H_{2,p}^{m+1}[a, b] \cap S_{2m}(\Delta_n) = \{S_T\}$$

and the functional $J_a: H_{2,p}^{m+1}[a, b] \rightarrow \mathbb{R}_1$ defined by

$$(17) \quad J_a(g) = \int_a^b [g^{(m+1)}(x)]^2 dx$$

attains its minimum at the function S_T :

$$(18) \quad \min \{J_a(g) : g \in H_{2,p}^{m+1}[a, b]\} = J_a(S_T)$$

Also, the inequality

$$(19) \quad \|g^{(m+1)} - S_T^{(m+1)}\|_2 \leq \|g^{(m+1)} - S^{(m+1)}\|_2$$

holds for any $S \in S_{2m}(\Delta_m)$.

The relations (18) and (19) (called "the minimal norm property" and "the best approximation property", respectively) are proved in [2], following a way similar to that used to prove the corresponding properties for interpolating natural spline functions (see [6], Theorems 3 and 4).

We mention that the derivative-interpolating spline functions of order $2m$ have been successfully used for the numerical solution of boundary value problems (Cauchy problems) for differential equations with modified argument ([6]). Spline functions of degree 5 (particular cases of p -derivative-interpolating spline functions for $p = 2$ and $m = 2$) were used in [8] to solve a singularly perturbed bilocal problem.

In the next we shall show that the functions used in [8] are spline functions obtained by integrating the interpolation natural cubic spline functions.

Lemma 3 Let $s \in S_{2m-1}(\Delta_n)$, $\alpha \in [a, b]$ fixed and

$$(20) \quad \hat{I}(s) := \left\{ \int_\alpha^x s(t) dt + C : C \in \mathbb{R} \right\}.$$

Then every $S \in \hat{I}(s)$ belongs to $S_{2m}(\Delta_n)$.

Proof. By (2)

$$s(x) = \sum_{i=0}^{m-1} B_i x_i + \sum_{k=1}^n b_k (x - x_k)_{+}^{2m-1},$$

with

$$\sum_{k=1}^n b_k x_k^j = 0, \quad j = 0, 1, 2, \dots, m-1.$$

Consequently

$$\begin{aligned} S(x) &= \int_a^x s(t) dt + C_0 = C_0 + \sum_{i=0}^{m-1} \frac{B_i}{i+1} x^{i+1} + \sum_{k=1}^n \frac{b_k}{2m} (x - x_k)_{+}^{2m} = \\ &= \sum_{i=0}^m A_i x^i + \sum_{k=1}^n a_k (x - x_k)_{+}^{2m} \end{aligned}$$

where $A_0 = C_0$, $C_1 = C_0 + \sum_{i=0}^{m-1} \frac{B_i}{i+1} \alpha^{i+1} + \sum_{k=1}^n \frac{b_k}{2m} (\alpha - x_k)_{+}^{2m}$, $A_i = \frac{B_i}{i+1}$,

$i = 1, 2, \dots, m$; $a_k = \frac{b_k}{2m}$, $k = 1, 2, \dots, n$ and $\sum_{k=1}^n a_k x_k^j = 0$, $j = 0, 1, \dots, m-1$.

Taking into account (12) and (13) it follows $S \in S_{2m}(\Delta_n)$. $\square$

Lemma 4 Let $f \in H_2^m[a, b]$ and

$$(21) \quad \hat{I}(f) : \left\{ \int_a^x f(t) dt + c : c \in \mathbb{R} \right\}$$

Then $g \in \hat{I}(f)$ if and only if $g \in H_2^{m+1}[a, b]$.

Proof. Obviously that $\hat{I}(f) \subset C^m[a, b]$, and if $g \in \hat{I}(f)$ then $g^{(m)} = f^{(m-1)}$ (absolutely continuous on $[a, b]$) and $g^{(m+1)} = f^{(m)} \in L_2[a, b]$, showing that $g \in H_2^{m+1}[a, b]$.

If $g \in H_2^{m+1}[a, b]$ then $g' \in H_2^m[a, b]$ so that

$$g(x) = \int_a^x g'(t) dt + g(a)$$

i.e. $g \in \hat{I}(g')$. $\square$

Lemma 5 Let $Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and $\overline{Y} = (y_0, Y) = (y_0, y_1, y_2, \dots, y_n) \in \mathbb{R}^{n+1}$. Then the operator

$$I_\alpha : H_{2Y}^m[a, b] \rightarrow H_{2\overline{Y}}^{m+1}[a, b]$$

defined by

$$(22) \quad I_\alpha(f)(x) = \int_a^x f(t) dt + y_0, \quad x \in [a, b]$$

is bijective.

Proof. Obviously that

$$\begin{aligned} I_\alpha(f)(x_i) &= f(x_i) + y_i, \quad i = 1, 2, \dots, n \\ I_\alpha(f)(a) &= y_0 \end{aligned}$$

showing that $I_\alpha(f) \in H_{2\overline{Y}}^{m+1}[a, b]$, for every $f \in H_{2Y}^m[a, b]$.

If $f_1, f_2 \in H_{2Y}^{m+1}[a, b]$ and $I_\alpha(f_1) = I_\alpha(f_2)$ then

$$\int_a^x f_1(t) dt = \int_a^x f_2(t) dt$$

for all $x \in [a, b]$, implying $f_1(t) = f_2(t)$ for all $t \in [a, b]$, i.e.

I_α is injective.

Let $g \in H_{2\overline{Y}}^{m+1}[a, b]$. Then $g' \in H_{2Y}^m[a, b]$ and $I_\alpha(f) = g$, for $f = g'$, showing that I_α is surjective, too. $\square$

Lemma 6 $I_\alpha(s_Y) = S_{\overline{Y}}$

Proof. By Lemmas 1 and 2,

$$I_\alpha(s_Y) \in S_{\Delta_n}(\Delta_n) \cap H_{2\overline{Y}}^{m+1}(\Delta_n) = \{S_{\overline{Y}}\}$$

so that

$$I_\alpha(s_Y) = S_{\overline{Y}}. \quad \square$$

Lemma 7 $J_\alpha = J \circ I_\alpha^{-1}$

Proof. For $g \in H_{2,Y}^{m+1}[a,b]$ we have

$$\begin{aligned} J_a(g) &= \int_a^b \left| g^{(n+1)}(x) \right|^2 dx = \int_a^b \left| (g')^{(n)}(x) \right|^2 dx = \\ &= \int_a^b \left| (I_a^{-1}(g)'(x))^{(n)} \right|^2 dx = J(I_a^{-1}(g)) = (J \circ I_a^{-1})(g) \quad \square \end{aligned}$$

Theorem 8 a) If the functional $J : H_{2,Y}^n[a,b] \rightarrow \mathbb{R}_+$ attains its minimal value at the spline function $s_Y \in H_{2,Y}^n[a,b] \cap S_{2m-1}(\Delta_n)$ then the functional $J_a : H_{2,Y}^{n+1}[a,b] \rightarrow \mathbb{R}_+$ attains its minimal value at $S_Y \in H_{2,Y}^{n+1}[a,b] \cap S_{2m}(\Delta_n)$;

b) If $f \in H_{2,Y}^n[a,b]$ and

$$\|f^{(n)} - s_Y^{(n)}\|_2 \leq \|f^{(n)} - s^{(n)}\|_2,$$

for any $s \in S_{2m-1}(\Delta_n)$ then

$$\|I_a^{(n+1)}(f) - S_Y^{(n+1)}\|_2 \leq \|I_a^{(n+1)}(f) - S^{(n+1)}\|_2,$$

for any $S \in S_{2m}(\Delta_n)$.

Proof. a) For $g \in H_{2,Y}^{n+1}[a,b]$ we have

$$\|g^{(n+1)}\|_2^2 - \|S_Y^{(n+1)}\|_2^2 = \|(g')^{(n)}\|_2^2 - \|s_Y^{(n)}\|_2^2 \geq 0,$$

because $g' \in H_{2,Y}^n$. Also $\|f^{(n)}\|_2^2 \geq \|s_Y^{(n)}\|_2^2$ for any $f \in H_{2,Y}^n$ (see (7)). Therefore

$$\min \{ J_a(g) : g \in H_{2,Y}^{n+1} \} = J_a(S_Y).$$

b)

$$\begin{aligned} \|I_a^{(n+1)}(f) - S_Y^{(n+1)}\|_2 &= \|(I_a^{(n)}(f) - s_Y^{(n)})'\|_2 = \|f^{(n)} - s_Y^{(n)}\|_2 \leq \\ &\leq \|f^{(n)} - s^{(n)}\|_2, \end{aligned}$$

for any $s \in S_{2m-1}(\Delta_n)$. Since, by Lemma 1,

$$S(x) = \int_a^x s(t) dt + C \in S_{2m}(\Delta_n)$$

it follows $S^{(m)} = S^{(m+1)}$ and $f^{(m)} = (I_n(f))^{(m+1)}$ so that

$$\|f^{(m+1)}(f) - S^{(m+1)}\|_2 \leq \|(I_n(f))^{(m+1)} - S^{(m+1)}\|_2$$

for any $S \in S_{2m}(\Delta_n)$. ■

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"T.Popoviciu" Institute
of Numerical Analysis
str. Republicii Nr. 37
3400 Cluj-Napoca
Romania