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Dedicated to Professor Iulian Coroian on his 60th anniversary

On the efficiency of the computations for approximating the solutions of equations

Ion Păvăloiu

1 Introduction

It is well known that the Lagrange-Hermite type inverse interpolation offers a lot of iterative methods for solving equations. The preference for one method or for another in obtaining a suitable approximation of a solution must take into account, among other facts, not only the convergence order of the chosen method, but also the amount of calculations needed at each iteration step. While the convergence order may be determined exactly, the amount of calculations from each iteration may be difficult to determine exactly.

The problem we are interested in this note consists in considering a certain class of iterative methods and in determining the methods of that class which have an optimal efficiency index.

Concerning the efficiency index, we shall adopt the definition given by A.M. Ostrowski in [6]. As we shall see in the following, the essential elements in defining the efficiency index of an iterative method are the convergence order and the number of function evaluations needed at each iteration step. As it is well known, the Lagrange-Hermite type interpolatory methods use the values of the considered function, as well as the values of its derivatives up to a certain order. The efficiency index introduced by Ostrowski, as we shall see, seem to be well suited to our purposes.

We shall restrict here by considering the class of methods given by the Chebyshev-type methods, which was also studied in [8], [9].

We shall show that the results obtained in [8] and [9] remain valid under more general hypotheses than those considered in those papers. More exactly, we shall admit here that the amount of calculations need at each iteration step is proportional with the number of functions whose values are needed. The factor of proportionality $\delta > 0$ is considered constant for a given function.

Let $I = [a, b]$, $a, b \in \mathbb{R}$, $a < b$ be an interval of the real axis and consider the equation

$$f(x) = 0 \tag{1}$$

where $f : I \to \mathbb{R}$. We assume that equation (1) has a unique solution $\overline{x} \in]a, b[$. For solving this equation we consider the function $g : I \to I$ and we suppose that $\overline{x}$ is a fixed point for g. We also assume that the sequence $(x_p)_{p \geq 0}$ given by the iterations

$$x_{p+1} = g(x_p),\ p = 0, 1, \ldots,\ x_0 \in I, \tag{2}$$

converges. Obviously, if g is continuous at $\overline{x}$, then $\lim\limits_{p\to\infty} x_p = \overline{x}$.

Concerning the function f, we shall also assume the following:

a) the function f is differentiable at $\overline{x}$;

b) for any $x, y \in I$, we have that $0 < [x, y; f] < m$, for some $m > 0$, where $[x, y; f]$ is the divided difference of f on the points x and y.

Definition 1 *The sequence $(x_p)_{p\geq 0}$ has the convergence order $\omega \geq 1$, if there exists the limit*

$$\alpha = \lim_{p\to\infty} \frac{\ln |g(x_p) - \overline{x}|}{\ln |x_p - \overline{x}|}$$

and $\alpha = \omega$.

The following lemma can be easily proved.

Lemma 2 *If the function f satisfies a) and b), then the necessary and sufficient condition for the sequence $(x_p)_{p\geq 0}$ given by (2) to have the convergence order $\omega \in \mathbb{R}$, $\omega \geq 1$, is that there exists the limit*

$$\beta = \lim_{p\to\infty} \frac{\ln |f(g(x_p))|}{\ln |f(x_p)|}$$

and $\beta = \omega$.

We shall denote in the following by m_p the number of functions whose values must be computed at each step p in the iterations (2).

Taking into account lemma 1 and according to the efficiency index proposed by Ostrowski, we shall admit the following definition.

Definition 3 *The number $E \in \mathbb{R}$ is called the efficiency index of the method (2) if there exists the limit*

$$l = \lim \left(\frac{\ln |f(g(x_p))|}{\ln |f(x_p)|} \right)^{\frac{1}{m_p}}$$

and $l = E$.

Remark 1 *If the sequence $(x_p)_{p \geq 0}$ given by (2) has the convergence order ω and there exists $p_0 \in \mathbb{N}$ such that for $p > p_0$ we have $m_p = r$, $r =$ constant, then*

(3) $$E = \omega^{\frac{1}{r}}$$

2 Chebyshev-type iterative methods

Let $q \in \mathbb{N}$ be a natural number, and assume that the function f obeys

α) the function f is q times differentiable on $]a, b[$.

β) the derivative satisfies $f'(x) \neq 0$, for any $x \in]a, b[$.

In the above hypotheses it follows that the function f admits an inverse $f^{-1} : D \to I$, where $D = f(I)$, and the solution $\overline{x}$ of (1) verifies

(4) $$\overline{x} = f^{-1}(0).$$

Moreover, the function f^{-1} is q times differentiable on each point from the interior of D, and the k-th derivative, where $1 \leq k \leq q$, is given by

(5) $$\left[f^{-1}(y)\right]^{(k)} =$$
$$\sum \frac{(2k - i_1 - 2)!(-1)^{k+i_1-1}}{i_2! i_3! \dots i_k!} \left[f'(x)\right]^{i_1 - 2k + 1} \left(\frac{f''(x)}{2!}\right)^{i_2} \left(\frac{f'''(x)}{3!}\right)^{i_3} \cdots \left(\frac{f^{(k)}(x)}{k!}\right)^{i_k}$$

where $y = f(x)$, and the above sum extends on all nonnegative integer solutions of the system

$$\begin{aligned} i_2 + 2i_3 + \cdots + (k-1)\, i_k &= k - 1 \\ i_1 + i_2 + \cdots + i_k &= k - 1. \end{aligned}$$

Let $x_0 \in I$ be given. The class of Chebyshev-type iterative methods is given by

(6) $$x_{p+1} = x_p + \sum_{s=1}^{q-1} (-1)^s \frac{\left[f^{-1}(y_p)\right]^{(s)}}{s!} f^s(x_p), \; p = 0, 1, \ldots,$$

where $y_p = f(x_p)$.

Remark 2 *For $q=2$, relation (6) yields the Newton method, i.e.*

$$x_{p+1}=x_p-\frac{f(x_p)}{f'(x_p)},\quad p=0,1,\ldots.$$

For $q=3$, one gets the Chebyshev method of order 3:

$$x_{p+1}=x_p-\frac{f(x_p)}{f'(x_p)}-\frac{1}{2}\frac{f''(x_p)f^2(x_p)}{(f'(x_p))^3},\ p=0,1,\ldots, \tag{7}$$

For different values of q, relation (6) generates a class of iterative methods of Chebyshev type. We shall denote by (CM) this class.

Remark 3 *It can be easily shown that for any fixed q, the convergence order of the corresponding method is q.*

Remark 4 *For a fixed q, at each iteration step in (6) there is necessary, by (5), to evaluate*

$$f,f',\ldots,f^{(q-1)}$$

at x_p, i.e. there are needed q function evaluations.

But (6) implies that are also needed:

- the values of the successive derivatives of f up to the $q-1$-th order;

- the values of the successive derivatives of f^{-1}, up to the $q-1$-th order

- one extra function evaluation, given by (6).

For these reasons, we shall admit that for the class (CM) there are needed δq function evaluations at each step, where the constant $\delta>0$ is given for any considered function f.

3 The optimal efficiency index.

Taking into account remarks 3 and 4, by (3) it follows the following expression for the efficiency index of the class (CM):

$$E=\varphi(q)=q^{\frac{1}{\delta q}}.$$

Considering the function $\varphi(t)=t^{\frac{1}{\delta t}}$, $\varphi:(0,\infty)\to\mathbb{R}$, there can be easily verified that it attains its maximum value at the unique point $t=e$. It is clear then that $E=\varphi(q)$, $q\in\mathbb{N}$ has the maximum value at $q=3$. The following result holds.

Theorem 4 *Considering the class (CM), the method with the highest efficiency index is the Chebyshev method (7).*

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"T. Popoviciu" Institute
of Numerical Analysis
str. Republicii Nr.37
P.O. Box 68
3400 Cluj-Napoca
ROMANIA