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Dedicated to professor Iulian Coroian on his 60th anniversary

THE EXISTENCE OF THE SOLUTION FOR THE EQUATION MODELLING THE ELASTIC CONTACT PROBLEM

Nicolae POP

Abstract. The aim of this article is to demonstrate the existence of the solution of the equation modelling the problem of elastic contact using the Brezis' fundamental theorem of pseudo-monotonous operators [4].

I. THE DIFFERENTIAL AND THE VARIATIONAL FORMS OF THE CONTACT PROBLEM

Let us consider two elastic bodies which, at the moment t = 0 are located within disjunctive domains $\Omega^1 \subset \mathbb{R}^d$ and $\Omega^2 \subset \mathbb{R}^d$ where d=2 or d=3. The boundaries of the bodies are giving by

$$\partial\Omega^1 = \Gamma^1 = \overline{\Gamma}^1_U \cup \overline{\Gamma}^1_N \cup \overline{\Gamma}^1_C \quad \text{and} \quad \partial\Omega^2 = \Gamma^2 = \overline{\Gamma}^2_U \cup \overline{\Gamma}^2_N \cup \overline{\Gamma}^2_C ,$$

which are open from a topological point of view, and disjunctive two by two, such that only Γ_C^1 and Γ_C^2 can have common points. The boundary values, in displacement and under stress are given by $\overline{u}(t,x)$ on the boundary $\Gamma_U = \Gamma^1_U \cup \Gamma^2_U$ and by $\overline{h}(t,x)$ on the boundary $\Gamma_N = \Gamma_N^1 \cup \Gamma_N^2$, respectively.

The boundary of $\Gamma_C = \Gamma_C^1 \cup \Gamma_C^2$ is considered at the beginning as being under no stress. At the same time we define the vector $\sigma^{(n)}(u)$ oriented outwards the boundary $\partial\Omega = \partial\Omega^1 \cup \partial\Omega^2$. We also know the initial displacements $u(0,x) = u_0(x)$ and the initial speed $\dot{u}(0,x) = u_1(x)$. As long as the two bodies do not touch each other, the field of displacement will be the solution of a boundary value problem for the partial differential equations of elasto-dynamics. If the bodies touch each other, then in the contact areas there are forces preventing their mutual penetrating. The boundary condition which has to be formulated in this latter case is called "*contact condition*". On the contact area there may appear, additional, friction forces, as well, which will be described by a friction law.

The contact problem of two elastic bodies on a time interval $[0, t_E]$, $t_E > 0$ has the following differential form:

(1.1) $\rho \ddot{u}_i(t,x) - \sigma_{ij,j}(u(t,x)) = f_i(t,x), \quad [0,t_E] \times \Omega$

purceding in $[0,t_E] \times \Omega$, where $\Omega = \Omega^1 \cup \Omega^2$. The boundary conditions are

(1.2) $u(t,x) = \overline{u}(t,x)$ on $[0,t_E] \times \Gamma_U$

and

(1.3) $\sigma^{(n)}(u)(t,x) = \bar{h}(t,x)$, on $[0, t_1] \times \Gamma_N$

The initial conditions are

(1.4) $u(0,x) = u_0(x)$ *and* $\dot{u}(0,x) = u_1(x)$.

Contact condition have to contain both the condition of non-penetrating of one body into another (or the penetrating according to a given law), and the correct description of the transmission of forces among the bodies. These processes must be formulated correctly from a mathematical point of view in such a way that they may be approached through variation methods.

Because of inherent difficulties *"the contact condition"* is approximated by the Signorini condition [3]. In approximating "the contact condition" ammounts to account for the conditions of the linear elasticity theory. We will begin by parametering the two contact boundaries Γ_C^1 and Γ_C^2, which are assumed to be disjunctive. To this goal,we shall resort to two bijective applications:

$x^{(1)}: P \to \Gamma_C^1$ and $x^{(2)}: P \to \Gamma_C^2$

of a domain P of a C^1 class whose dimension is d-1 for each contact boundary. Consequently in each point $x \in P$ the are deals with:

- the normalized normal vector on Γ_C^1 :

(1.5) $n(x) := \dfrac{x^{(2)}(x) - x^{(1)}(x)}{\left|x^{(2)}(x) - x^{(1)}(x)\right|}$,

the initial gap

(1.6) $g(x) := \left|x^{(2)}(x) - x^{(1)}(x)\right|$.

For a u-displacement field on $\overline{\Omega^1} \cup \overline{\Omega^2}$ we define the relative displacement

(1.7) $u^R(x) := u^{(1)}(x^{(1)}(x)) - u^{(2)}(x^{(2)}(x))$

where $u^{(j)} = u_{|\partial\Omega^j}$ with j=1,2 , denotes the trace of u on $\partial\Omega$.

The components of v vector field $v: P \to \mathbb{R}^d$, in the direction of n (perpendicular to n) are denoted by $v_N := v{\cdot}n$ (respectively, $v_T = v - v_N$).The condition of non-penetrating bodies will be understood as a geometric "contact condition". It is approximated by the inequation

(1.8) $u_N^R(x,t) \le g(x)$

This inequation describes actually *"the contact condition"* if the points on the contact boundaries move in the direction $n(x)$.

The "*contact condition*" has also describe safely the transmission of forces between the bodies and to fulfil the following requirements:

1°. The Newton's balance of forces , i.e. the force F^{12} which is exercised by the body Ω^1 on the body Ω^2 , is oposite to force F^{21} which Ω^2 exercises on Ω^1;

2°. On the contact area there can be transmitted only compressive forces;

3°. The forces can be transmitted only in the areas where the bodies touch

Condition 1° means:

(1.9) $$\sigma^{(n)}(x^{(1)}(x))\, J_1(x) = -\sigma^{(n)}(x^{(2)}(x))\, J_2(x) =: \sigma(x) \qquad \forall x \in P,$$

where $J_k(x)$, k=1,2 are the common determinants of the parameters

Condition 2° can be formulated as follows:

(1.10) $$\sigma_N(x) \le 0, \quad \forall x \in P,$$

whereas condition 3° leads to

(1.11) $$\sigma_N(x)\,(u_N^R(x) - g(x)) = 0 \;\; \forall x \in P$$

To summarise, the contact condition can be modelled as follows:

(1.12) $$(\sigma^{(n)} \circ x^{(1)})\, J_1 = -(\sigma^{(n)} \circ x^{(2)})\, J_2 =: \sigma$$

and

(1.13) $$u_N^R \le g\ ;\ \sigma_n \le 0\ ;\ \sigma_n(u_N^R - g) = 0$$

The variational form of Eq. (1.13) is

(1.13') $$u_N^R \le g \quad \sigma_N(v_N^R - u_N^R) \ge 0 \;\; \forall\; v_N^R \le g$$

The friction law that describes the dependence of the tangential stress on the normal stress as weel as on the sliding speed is given by

(1.14a) $$\dot u_T^R = 0 \Rightarrow |\sigma_T| \le \mathcal{F}(0) \cdot |\sigma_N|,$$

and

(1.14 b) $$\dot u_T^R \ne 0 \Rightarrow \sigma_T = -\mathcal{F}(\dot u_T^R) \cdot |\sigma_N| \cdot \frac{\dot u_T^R}{|\dot u_T^R|}$$

where $\mathcal{F}$ describs the friction coefficient depending on the speed $\dot u_T^R(x)$ with which the bodies slide one to an other in the point x.

The differential (classical) formulation of the dynamical problem consists in: *finding the solution of the system of differential equations system:*

$$\rho(x)\, \ddot u_i(t,x) - \sigma_{ij,j}(u(t,x)) = f_i(t,x) \quad in\ [0,t_E] \times \Omega$$

with the boundaries values $u(t,x) = \bar u(t,x), \quad on\ [0,t_E] \times \Gamma_U,$

$$\sigma^{(n)}(u(t,x)) = \bar h(t,x), \quad on\ [0,t_E] \times \Gamma_N$$

$$\left.\begin{array}{l}(\sigma^{(n)} \circ x^{(1)})\, J_1 = -(\sigma^{(n)} \circ x^{(2)})\, J_2 =: \sigma \\ u_T^R \le g\ ;\ \sigma_N = 0\ ;\ \sigma_N(u_T^R - g) = 0 \\ \dot u_T^R = 0 \Rightarrow |\sigma_T| \le \mathcal{F}(0) \cdot |\sigma_N| \\ \dot u_T^R \ne 0 \Rightarrow \sigma_T = -\mathcal{F}(\dot u_T^R) \cdot |\sigma_N| \cdot \dfrac{\dot u_T^R}{|\dot u_T^R|},\end{array}\right\} \; on\ [0,t_E] \times P,$$

and the initial conditions.

$$u(x,0) = u_0(x), \; \dot{u}(0,x) = u_1(x)$$

A correct physical interpretation of the static contact problem is possible only if this is considered as an incremental step of a temporal discretization of the problem.

The initial problem will be equivalent with the iterrative determination of the displacement u at a given moment t, after the approximation of the temporal derivatives $\dot{u}$ and , $\ddot{u}$ by finite differences.To simplify the solution of this static contact problem, we shall transformit in a bonday-value problem and in a contact-problem with homogenous boundary values, except on the contact boundary. Here we shall restrict ourselves to the second problem.

In the case of variational formulation, the stress on the contact boundaries are defined as functions on Sobolev spaces as: $H^{-1/2}(\Gamma_C^1, \mathbb{R}^d) \times H^{-1/2}(\Gamma_C^2, \mathbb{R}^d)$, $f \in H_0^{-1}(\Omega^1, \mathbb{R}^d) \times H_0^{-1}(\Omega^2, \mathbb{R}^d)$, $\overline{u} \in H^1(\Omega^1, \mathbb{R}^d) \times H^1(\Omega^2, \mathbb{R}^d)$ and

$$\overline{h} \in \overline{H}^{-1/2}(\Gamma_N^1, \mathbb{R}^d) \times \overline{H}^{-1/2}(\Gamma_N^2, \mathbb{R}^d).$$

The variational formulation of the contact problem with homogeneous boundary values, except on the contact boundary consists in *looking for* $u \in K$ *i.e.* $\forall v \in K$ *such shat*:

(1.15) $$<\rho\ddot{u}, v-u>_\Omega + a(u,v-u) + j(u,v) - j(u,u) \geq -<H_T, v_T^R - u_T^R>_P$$

where $a(u,v) = \int_\Omega \sigma_{ij}(u)\, e_{ij}(v)\, dx$ descibs the deforming energy, $j(u,v) = \int_P \mathcal{F}(u_T^R)\, |\sigma_N(u)| \cdot |v_T^R|\, ds$ is the functional which descibs the influence of friction, whereas H_T is the tangential stress obtained from the boundary value problem, and $K := \{v \in H^1(\Omega^1, \mathbb{R}^d) \times H^1(\Omega^2, \mathbb{R}^d) : v = 0 \quad on\ \Gamma_U \;\; \& \;\; v_N^R \leq g \quad on\ P\}$ is the admissible set of our functions.The contact condition can also be approximated by the method of penalization, in which case the penalized functional is

$$\Phi_\delta = \int_P \frac{1}{\delta}[u_N^R - g]_+ v_N^R\, ds\ .$$

The functional that models the friction reads:

$$j_\delta(u,v) = \int_P \mathcal{F}(u_T^R) \cdot \frac{1}{\delta}[u_N^R - g]_+ \cdot |v_T^R|\, ds_x\ .$$

The next step is to approximate the variational inequation (1.15) by a variational equation, i.e. to approximate the module $|v_T^R|$ from the friction functional with a differentiating convex function depending on a parameter ε , fulfilling the conditions $|\, grad\, \phi_\varepsilon(x) \,| \le 1$, $|\, \phi_\varepsilon(x) - |x| \,| < \varepsilon$, $(\forall)\, x \in \mathbb{R}^d$.

So, the variation equation that approximates the variation inequation (1.15) is given by:

(1.16) $$<\rho u, v>_\Omega + a(u,v) + \phi_\delta(u,v) + \psi_{\delta,\varepsilon}(u,v) = -<H_\tau, v_\tau^N>_\Gamma$$

where

$$\psi_{\delta,\varepsilon}(u,v) = \lim_{\lambda \to 0}(j_{\delta,\varepsilon}(u, u+\lambda v) - j_{\delta,\varepsilon}(u,u)) =$$
$$= \int_\Gamma \mathcal{F}(u_\tau^N)\frac{1}{\delta}[u_\tau^N - g]_+ \cdot grad\,\phi_\varepsilon(u_\tau^N) \cdot v_\tau^N\, ds_x$$

is the Coulomb friction law which describs in tourn as follows:

$$\sigma_\tau(u) = -\mathcal{F}\,|\,\sigma_N(u)\,| \cdot grad\,\phi_\varepsilon(u_\tau) \;.$$

This corresponds to the regularization of the friction law through the functions ϕ_ε

2. THE EXISTENCE OF A SOLUTION TO THE OPERATORIAL EQUATION MODELLING THE CONTACT PROBLEM

After the preliminary problems done above, we shall formulate the fundamental theorem for the pseudo-monotonous operators. To do this, let be X a real , reflexive and separable Banach space, $A : X \to X^*$ a nonlinear operator and $f \in X^*$ a given functional. Further $\{w_1, w_2, \ldots\}$ denotes a Galerkin basis, which is a sub-set of the set of linearly independent elements fulfilling the condition: $\overline{\bigcup_{k=1}^{\infty} span\{w_1, \ldots, w_k\}} = X$ where $X_k = span\{w_1, \ldots, w_k\}$ is the space generated by the first k vectors of the basis. We will analyze the operatorial equation

(1.17) $$Au = f$$

where $u \in X$ as well as the corresponding Galerkin equation:

(1.18) $<A u_k, w_j> = <f, w_j>$, $j = 1, \ldots, k$ for which $u_k = \sum_{i=1}^{k} C_i^{(k)} w_i \in X_k$.

Theorem 2.1 (*The fundomental Brezis theorem of the pseudo-monotonous operators [4]*).

Let be X a real, reflexive and separable Banach space, and $A : X \to X^*$ *a* *pseudo-monotonous, continuous and coercive operator. Then for* $\forall f \in X^*$:

1. The Garlekin equation(1.18) has at least one solution u_k, $(\forall)\ k \in \mathbb{N}$.

2. If $(u_{k'})_{k'}$, *is a weakly convergent sub-row of the row of solutions* $(u_k)_k$ *of point 1, then the weak value is a solution of the operatorial equation (1.1"), as shown in [4].*

The next step consists in demonstrating that the equation (1.16) verifies the conditions of theorem 2.1. For this purpose, we will define the operators:

$$<Pu,v> = <\rho u,v>_\Omega + a(u,u),\quad (\forall)\, u,v \in V;$$
$$<Qu,v> = \phi_\delta(u,v);\quad \text{and } <Ru,v> = \psi_{\delta,\varepsilon}(u,v),\quad (\forall)\ u,v \in V.$$

This opens the way to our Lemma 2.1:

Lemma 2.1. If X is a reflexive Banach space, $A:X\to X^*$ is a strong monotonous operator and $B:X\to X^*$ is a complete continuous operator, then the operator $T:=A+B$ is pseudo-monotonous [1].The operators P, Q and R have the following qualities:

Lemma 2.2 Under lemma 2.1 and the theorem 2.1, we have:

1. $P:V\to V^*$ is a linear, continuous and elliptical operator;
2. $Q:V\to V^*$ is a continuous, monotonous and Lipschitz continuous operator;
3. $R:V\to V^*$ is a completely continuous operator.

The demonstration can be found in [2].With these results, according to lemma 2.1, the operator $A:=P+Q+R$ fulfils the condition of theorem 2.1 concerning pseudo-monotonous operators, so that A is pseudo-monotonous, continuous and coercive operator. This show the existence of the solution of the operatorial equation $Au=f$.

REFERENCES

[1] **C. Eck,** Existenz und Regularität der Lösungen für Kontaktprobleme mit Reibung, Disertation, Univ. Stuttgart, 1996

[2] **N. Pop,** Applications of the Variational Inequalities in the Elastic Contact Problems, CUB PRESS 22, Baia Mare, 1998

[3] **A. Signorini,** Sopra alune questioni di elastostatica, *Atti. Soc. Ital. Progr. Sci.*, 1933.

[4] **E .Zeidler,** Nonlinear Functional Analysis and its Applications II B: Nonlinear Monotone Operators, Springer-Verlag, New-York-Berlin-Heidelberg-Tokio, 1990

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North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 4800 Baia Mare
ROMANIA, E-mail: nicpop@univer.ubm.ro