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Dedicated to Professor Iulian Coroian on his 60th anniversary

REDEFINING n-MODULES

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Abstract

We give a more general definition of (left) R-n-modules by omitting the condition of uniqueness of the neutral element of the abelian n-group involved (this condition initially appeared in [1], and was always used in further papers concerning n-modules). We give some examples and properties of n-submodules, homomorphisms, congruences and factor n-modules. A class of special automorphisms and one of n-submodules are introduced. A correspondence between n-submodules of the factor n-module and certain n-submodules of the initial n-module is established.

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1 Introduction

In 1980 N. Celakoski [1] defined n-modules, over an associative and unital ring as a generalization of the usual concept of modules. Some further results concerning this new structure and its binary reducts as well as some categorical properties were established in [1],[4],[8],[7],[3]. However, in [1] a

strong restriction was imposed: the abelian n-group employed has a *unique* neutral element.
In this paper we use the same set of axioms in order to define n-modules, but we omit the above condition on unique neutral element. In this way, the structure introduced by Celakoski in [1] becomes a particular case of what we shall call n-modules with zero.

A left R-n-module, where $n \geq 2$ and R is an arbitrary associative and unital ring, is an abelian n-group M together with a mapping $\mu : R \times M \to M$ satisfying a certain set of axioms (the n-ary analogous of the ones employed in the binary case).
Homomorphisms between R-n-modules (R and n are fixed) are the obvious ones, as are the notions of n-submodules, congruences, factor n-modules, simplicity, etc.
For notation and terminology see [1],[2],[3],[5]; we just recall that in an n-ary sum we denote k successive terms $x_1, \ldots, x_k$ by x_1^k and if they are all equal to x then we denote the sequence by $\overset{(k)}{x}$. We will denote by $\bar{x}$ the querelement (or the skew element) of x; it has the properties:

$$\left[\overset{(i-1)}{x}, \bar{x}, \overset{(n-i)}{x}\right]_+ = x \text{ for any } i = 1, \ldots, n;$$

$$\left[a, \overset{(i-1)}{x}, \bar{x}, \overset{(n-i-1)}{x}\right]_+ = \left[\overset{(i-1)}{x}, \bar{x}, \overset{(n-i-1)}{x}, a\right]_+ = a, \text{ for any } i = 1, \ldots, n-1.$$

We will also use the notation $x^{<k>}$ for the k-th power defined in the sense of Dörnte (see [2],[5]).

2 Redefining n-modules

In the sequel we shall denote by R an associative, unital ring ($1 \neq 0$).

Definition 2.1 A left R-n-module is an abelian n-group $(M, [\,]_+)$ together with a mapping $\mu : R \times M \to M$, which satisfies the following set of conditions:

1) $\mu(r, [x_1^n]_+) = [\mu(r, x_1), \ldots, \mu(r, x_n)]_+$

2) $\mu(r_1 + \ldots + r_n, x) = [\mu(r_1, x), \ldots, \mu(r_n, x)]_+$

3) $\mu(r \cdot r', x) = \mu(r, \mu(r', x))$

4) $\mu(1, x) = x$

for all $x, x_1, \ldots, x_n \in M$, $r, r', r_1, \ldots, r_n \in R$.

Right R-n-modules can be defined in the obvious way: replace condition 3) above by 3') $\mu(rr', x) = \mu(r', \mu(r, x))$. As in the binary case, the general study of right n-modules can be deduced from the study of left n-modules and conversely; for this reason we shall deal with left R-n-modules only and refer to them as to R-n-modules.

In order to simplify notation, we shall denote $\mu(r, x)$ by rx (or, sometimes, by $r * x$, $r \cdot x$, $r \circ x$) and call this the multiplication with the scalar r.

Recall that an element e of an n-group $(M, [\ ],)$ is called a *neutral element* of M if every n-sum of $n-1$ terms e and one term x equals x, for any $x \in M$. In an abelian n-group M an element $e \in M$ is a neutral element if and only if e is an idempotent.

Let us make now the following remarks: the empty n-group $\emptyset$ may be regarded as an R-n-module, for any associative and unital ring R; if M is a non-empty R-n-module, then it necessarily has (at least) one neutral element (indeed, for every $x \in M$, the element $0x$ is an idempotent, hence a neutral element);

let us denote by $\mathcal{N}$ (or sometimes by $\mathcal{N}_M$) the set of all neutral elements of the n-group M and define the subset $\mathcal{N}_0 \subseteq \mathcal{N}$ as

$$\mathcal{N}_0 = \{e \in \mathcal{N} \mid e = 0x, \text{ for some } x \in M\}$$

Note that if $e \in \mathcal{N}$ then $re \in \mathcal{N}$, for every $r \in R$ and $e \in \mathcal{N}_0$ iff $re = e$, $\forall r \in R$. If the set $\mathcal{N}_0$ consists of exactly one element, then this will be called a *zero* of the R-n-module and we shall denote it by 0.

In particular, if an n-group M has a unique neutral element e, then e is a zero of any R-n-module defined on M.

Examples 2.2 1) Let M be a one-element set. There is a unique n-group structure on M. Any ring R determines on this n-group a unique R-n-module called the zero R-n-module.

2) Let $(M, [\ ],)$ be an abelian n-group with $\mathcal{N}_M \neq \emptyset$ and let e be an arbitrary, fixed element of $\mathcal{N}_M$. We shall define a *standard* $\mathbb{Z}$-n-module on M, by using the external operation $\mu_e : \mathbb{Z} \times M \to M$, $\mu(k, x) = kx$ where $kx = \left[\overset{(q+1)}{x}, \overset{(n-2)(q+1)-r}{e} \right]_\cdot$, $k = (n-1)q + r$, $0 \leq r < n-1$. The element

e is a zero in this $\mathbb{Z}$-n-module. All the standard $\mathbb{Z}$-n-modules on M (defined by using all the neutral elements of M) are isomorphic.
3) Take an integer $p \geq 2$ and put $m = p(p+1)$, $n = p-2$. Define on $\mathbb{Z}_m$ the n-ary operation "$[\,]_+$" by: $[x_1^n]_+ = x_1 + \ldots + x_n$ (here "$+$" denotes addition modulo m). It is easily seen that $(\mathbb{Z}_m, [\,]_+)$ is an abelian n-group and its set of neutrals is $\mathcal{N} = \{0, p, 2p, \ldots, p^2\}$. Define the multiplication with scalars $\mu : \mathbb{Z} \times \mathbb{Z}_m \to \mathbb{Z}_m$, $\mu(k, x) = k * x$, $k * x = ((k-1)p + k) \cdot x$ (here "$\cdot$" denotes multiplication modulo m). It is easy to check that $(\mathbb{Z}_m, [\,]_+)$ together with the mapping μ is a $\mathbb{Z}$-n-module which has no zero element; in fact, $\mathcal{N}_0 = \mathcal{N}$. Indeed, we have $0 * x = -p \cdot x = p^2 - (x-1) \cdot p = (p+1-x) \cdot p$.
4) Take two integers $p, t \geq 1$ and put $m = pt(pt+1)$, $n = p^2 t + p + 1$. Define on $\mathbb{Z}_m$ the n-ary operation "$[\,]_+$" as before, by: $[x_1^n]_+ = x_1 + \ldots + x_n$, and the external operation $k \cdot x = (pt(k-1) + k) \cdot x$. As before, it is easy to check that $\mathbb{Z}_m$ together with this external operation is a $\mathbb{Z}$-n-module for which we have:

$$\mathcal{N} = \{0, t, 2t, \ldots, (p^2 t + p - 1)t\} \text{ and } \mathcal{N}_0 = \{0, pt, 2pt, \ldots, p^2 t^2\},$$

which is strictly included in $\mathcal{N}$, i.e. $\mathcal{N}_0 \subset \mathcal{N}$.

Proposition 2.3 *Let M be an R-n-module. Then for all $x \in M$, $r \in R$ we have*

$$r\bar{x} = \overline{rx},\ (-r)x = [0x, 0x, \overset{(n-3)}{rx}, r\bar{x}]_+;\ \bar{x} = (-n+2)x = ((-1) + \ldots + (-1))x;$$

if R is a division ring, then $rx \in \mathcal{N}$ implies $r = 0$ or $x \in \mathcal{N}$.

Proof. We have $rx = r[\overset{(n-1)}{x}, \bar{x}]_+ = [\overset{(n-1)}{rx}, r\bar{x}]_+$, which shows that $r\bar{x} = \overline{rx}$. In order to prove the next identity, note that $0x = ((-r) + r + 0 + \ldots + 0)x = [(-r)x, rx, \overset{(n-2)}{0x}]_+$. This implies that $(-r)x = [0x, 0x, \overset{(n-3)}{rx}, r\bar{x}]_+$. Finally,

$$(-n+2) = ((-1) + \ldots + (-1) + 0 + 0)x = [\overset{(n-2)}{(-1)x}, 0x, 0x]_+ =$$
$$[\overset{(n-2)}{0x}, \overset{(n-2)}{0x}, \overset{(n-2)(n-3)}{x}, \overset{(n-2)}{\bar{x}}, 0x, 0x]_+ = [\overset{(n-2)(n-3)}{x}, \overset{(n-3)}{\bar{x}}, \bar{x}]_+ = \bar{x}.$$

If $rx \in \mathcal{N}$ and $r \neq 0$ then $r^{-1}(rx) \in \mathcal{N}$, i.e. $x \in \mathcal{N}$. ∎

Definition 2.4 A subset S of an R-n-module M is an n-submodule of M if

1) $\forall x_1, \ldots, x_n \in S$, $[x_1^n]_+ \in S$
2) $\forall r \in R$, $\forall x \in S$, $rx \in S$.

Remarks 2.5 1) The empty subset $\emptyset$ is the smallest n-submodule of M and M itself is the biggest n-submodule of M. It is easy to check that the partially ordered set $(S_{Rn}(M), \subseteq)$ of all n-submodules of M is a complete lattice.
2) $\mathcal{N}$ and $\mathcal{N}_0$ are n-submodules of M.
3) $\{c\}$ is an n-submodule of M iff $c \in \mathcal{N}_0$.
4) For every non-empty n-submodule S of M we have $S \cap \mathcal{N}_0 \neq \emptyset$.
5) If S, T are two n-submodules of M, with $S \subseteq T$, then $\mathcal{N}_{0S} \subseteq \mathcal{N}_{0T}$ and $\mathcal{N}_S \subseteq \mathcal{N}_T$.
6) The notion of n-submodule generated by a subset $X \subseteq M$ is defined in the obvious way. We have: $\langle\emptyset\rangle = \emptyset$ and for $X \neq \emptyset$

$\langle X\rangle = $ [illegible]

Definition 2.6 Let M_1, M_2 be two R-n-modules. A map $f : M_1 \to M_2$ is called R-n-module homomorphism if:
$f([x_1^n]_+) = [f(x_1), \ldots, f(x_n)]_+$, for each $x_1, \ldots, x_n \in M_1$,
$f(rx) = rf(x)$, for each $r \in R$ and $x \in M_1$.

We shall denote by $\mathrm{Hom}_{Rn}(M_1, M_2)$ the set of all R-n-module homomorphisms from M_1 to M_2 and define an n-ary addition: $[f_1, \ldots, f_n]_+(x) = [f_1(x), \ldots, f_n(x)]_+$. It is easy to check that $\mathrm{Hom}_{Rn}(M_1, M_2)$ together with the addition defined above is an abelian n-group and its set of neuters is $\mathcal{N} = \{f \in \mathrm{Hom}_{Rn}(M_1, M_2) \mid f(M_1) \subseteq \mathcal{N}_2\}$. If the ring R is commutative, then $\mathrm{Hom}_{Rn}(M_1, M_2)$ is an R-n-module (the external operation being given by: $(rf)(x) = rf(x)$).

Proposition 2.7 *Let M_1, M_2 be two R-n-modules and $f \in \mathrm{Hom}_{Rn}(M_1, M_2)$. Then the following hold.*
1) $e \in N_1$ implies $f(e) \in N_2$ and $e \in N_{01}$ implies $f(e) \in N_{02}$.
2) $f(\overline{x}) = \overline{f(x)}$, for every $x \in M_1$.
3) $N \in S_{Rn}(M_1)$ implies $f(N) \in S_{Rn}(M_2)$ and $T \in S_{Rn}(M_2)$ implies $f^{-1}(T) \in S_{Rn}(M_1)$.

As immediate consequences we have: if M_1, M_2 are R-n-modules with zero then $f(0) = 0$; the set $\mathrm{Ker}\, f = \{x \in M_1 \mid f(x) \in N_{02}\} = f^{-1}(N_{02})$ is an n-submodule of M_1, $N_{01} \subset \mathrm{Ker}\, f$; the set $f(M_1)$ is an n-submodule of M_2.

Proposition 2.8 *Let M_1, M_2 be two R-n-modules and $f \in \mathrm{Hom}_{Rn}(M_1, M_2)$. Then the following hold.*
1) f is injective if and only if $\mathrm{Ker}\, f = N_{01}$ and $f_{|N_{01}}$ is injective.
2) f is surjective if and only if $f(M_1) = M_2$.
3) f is a monomorphism if and only if f is injective.
4) f is an epimorphism if and only if f is surjective.

Proof. 1) If f is injective, then clearly its restriction to N_{01} is injective. Let $x \in \mathrm{Ker}\, f$, i.e. $f(x) \in N_{02}$ then we have $rf(x) = f(x)$, $\forall r \in R$, or equivalently $f(rx) = f(x)$, $\forall r \in R$. Since f is injective, it follows that $rx = x, \forall r \in R$ which proves that $x \in N_{01}$.

Conversely, let us suppose that $\mathrm{Ker}\, f = N_{01}$ and $f_{|N_{01}}$ is injective. Let $x, y \in M_1$ with $f(x) = f(y)$. Then, for $e \in N_{01}$, we have:

$$[f(x), \overset{(n-3)}{f(y)}, \overline{f(y)}, f(e)]_n = f(e) \in N_{02} \text{ i.e. } f([x, \overset{(n-3)}{y}, \overline{y}, e]_n) \in N_{02},$$

so $[x, \overset{(n-3)}{y}, \overline{y}, e]_n \in \mathrm{Ker}\, f$. This implies that $[x, \overset{(n-3)}{y}, \overline{y}, e]_n \in N_{01}$; now since the restriction of f to N_{01} is injective, it follows that $[x, \overset{(n-3)}{y}, \overline{y}, e]_n = e$, which implies $x = y$.

3) Suppose f is a monomorphism and $e_1, e_2 \in N_{01}$ with $f(e_1) = f(e_2)$. Consider the maps $g, h: M \to M_1$, $g(x) = e_1$, $h(x) = e_2$, where M is an arbitrary R-n-module; obviously g and h are homomorphisms and $f \circ g = f \circ h$. Now, since f is a monomorphism, it follows that $g = h$, i.e. $e_1 = e_2$

Consider now the following homomorphisms g, h: Ker $f \to M$, $g(x) = 0x$, $h(x) = x$. For $x \in$ Ker f we have $f(x) \in N_0$ which implies $0f(x) = f(x)$, $\forall x \in$ Ker f. We have then

$$(f \circ g)(x) = f(0x) = f(x) = (f \circ h)(x),$$

hence $f \circ g = f \circ h$. Since f is a monomorphism it follows that $g = h$, i.e. $x = 0x$, $\forall x \in$ Ker f. Therefore Ker $f = N_0$. By using 1) it follows now that f is injective. ∎

Proposition 2.9 *Let $f: M_1 \to M_2$, $g: M_2 \to M_3$ be two R-n-module homomorphisms. Then $g \circ f$ is an R-n-module homomorphism, too.*

We will introduce now a special class of automorphisms and one of n-submodules of an R-n-module, which will play an important role in the study of n-modules.
Let M be an R-n-module and $e, f \in N_0$. Define the map $\varphi_{e,f}: M \to M$ by $\varphi_{e,f}(x) = [x, \overset{(n-2)}{e}, f]_+$; it is easily checked that $\varphi_{e,f} \in \mathrm{Aut}_{R_n} M$ and $\varphi_{e,f}(e) = f$. Moreover, $\varphi_{e,f}(x) \in N_0$ if and only if $x \in N_0$.

Proposition 2.10 *The automorphisms $\varphi_{e,f}$, where $e, f \in N_0$ have the following properties:*
1) For any $e, f, g \in N_0$ we have $\varphi_{e,f}(g) = \varphi_{e,g}(f)$.
2) Let $e \in N_0$ arbitrary, fixed. For any $f, g \in N_0$ we have $\varphi_{f,g} = \varphi_{e,\varphi_{f,g}(e)}$, which proves that the set $\Phi = \{\varphi_{f,g} \mid f, g \in N_0\}$ coincides with the set $\Phi_1 = \{\varphi_{e,f} \mid f \in N_0\}$.
3) Φ is a commutative normal subgroup of the group $(\mathrm{Aut}_{R_n} M, \circ)$.

Proof. 2) For any $x \in M$ we have:

$$\varphi_{f,g}(x) = [x, \overset{(n-2)}{f}, g]_+ = [x, \overset{(n-2)}{e}, e, \overset{(n-2)}{f}, g]_+ = \varphi_{e,\varphi_{f,g}(e)}(x).$$

3) It is easy to check that:

$$\varphi_{e,f} \circ \varphi_{e,g} = \varphi_{e,\varphi_{e,f}(g)} = \varphi_{e,g} \circ \varphi_{e,f};\ \varphi_{e,e} = \mathrm{id}_M;$$
$$\varphi_{e,f}^{-1} = \varphi_{e,\varphi_{f,e}(e)};\ \alpha^{-1} \circ \varphi_{e,f} \circ \alpha = \varphi_{\alpha^{-1}(e),\alpha^{-1}(f)}$$

for any $\alpha \in \mathrm{Aut}_{R_n} M$. ∎

Let M be an R-n-module and $e \in N_0$. The subset $M_e = \{x \in M \mid 0x = e\}$ is a non-empty n-submodule of M. Indeed $e \in M_e$; if $x \in M_e$, then $0(rx) = r(0x) = re = e$, which proves that $rx \in M_e$; if $x_1, \ldots, x_n \in M_e$, then $0[x_1^n]_+ = [0x_1, \ldots, 0x_n]_+ = [\overset{(n)}{e}]_+ = e$, so $[x_1^n]_+ \in M_e$. Let us remark that the set of all M_e, $e \in N_0$ form a partition of M, and that e is zero element in the n-module M_e.
It is also of interest the fact that for $\alpha \in \mathrm{Aut}\, M$, $\alpha(M_e) = M_{\alpha(e)}$, in particular $\varphi_{e,f}(M_e) = M_f$.

Definition 2.11 *Let M be an R-n-module and S a non-empty n-submodule of M. The set $M/S = \{x + (n-1)S \mid x \in M\}$, where*

$$x + (n-1)S = \{y \in M \mid \exists s_1, \ldots, s_{n-1} \in S \text{ such that } y = [x, s_1^{n-1}]_+\}$$

together with the operations $[x_1 + (n-1)S, \ldots, x_n + (n-1)S]_+ = [x_1^n]_+ + (n-1)S$; $r(x + (n-1)S) = rx + (n-1)S$, is an R-n-module called the factor n-module of M with respect to S.

Remarks 2.12 1) If $S = \{e\} \subset N_0$, then $M/S \cong M$.
2) $N_{0\,M/S} = \{e + (n-1)S \mid e \in N_0\}$.
3) The R-n-module M/S has a zero element if and only if $S \supseteq N_0$.
4) The natural map

$$p_S : M \to M/S,\ p_S(x) = x + (n-1)S$$

is a surjective n-module homomorphism.
5) $M/S = M/T$ if an only if there exists $e \in N_0$ such that $T = e + (n-1)S$.

Congruences on an R-n-module are defined in the obvious way; let us note that the lattice of congruences is modular.

The connections between congruences and n-submodules are described in the following

Proposition 2.13 *Let M be a R-n-module, S a non-empty n-submodule and ρ a congruence. Then:*
1) The binary relation ρ_S, defined by $x\rho_S y \Leftrightarrow \exists s_1, \ldots, s_{n-1} \in S$ such that $y = [x, s_1^{n-1}]_+$, is a congruence on M. Moreover, $M/S = M/\rho_S$.

2) The equivalence class $\rho(e)$ is an n-submodule of M, for each $e \in N_0$. Moreover $M/\rho = M/\rho(e)$.
3) A coset $x + (n-1)S$ is an n-submodule of M if and only if it contains at least one element of N_0.

Proof. 1. Since S is a non-empty n-submodule of M, there exists $e \in N_0 \cap S$, so we can write any $x \in M$ as $x = [x, e^{(n-1)}]_+$, which proves that ρ_S is reflexive. If $x \rho_S y$ then $y = [x, s_1^{n-1}]_+$ for some $s_1, \dots, s_{n-1} \in S$. Then $x = [y, [s_{n-1}^{(n-3)}, \bar{s}_{n-1}]_+, \dots, [s_2^{(n-3)}, \bar{s}_2]_+, [s_1^{(n-3)}, \bar{s}_1]_+]_+$, i.e. $y \rho_S x$. The relation ρ_S is clearly transitive. If $x_i \rho y_i$ for $i = 1, \dots, n$ then $y_i = [x_i, s_{i1}^{n-1}]_+$ for some $s_{i1}, \dots, s_{i,n-1} \in S$.
We obtain

$$[y_1^n]_+ = [x_1, s_{11}^{n-1}, \dots, x_n, s_{n1}^{n-1}]_+ =$$

$[[x_1^n]_+, [s_{11}^{n-1}, \dots, s_{n1}]_+, s_{\dots}^{\dots}]_+$ [illegible], which proves that $[x_1^n]_+ \rho_S [y_1^n]_+$.

2. If $x_1, \dots, x_n \in \rho(e)$, then $x_i \rho e$ for $i = 1, \dots, n$ and so $[x_1^n]_+ \rho [e^{(n)}]_+ = e$, as well as $x_1 \rho (rx) = e$, for all $r \in R$. This proves that $\rho(e)$ is an n-submodule of M.

3) Follows from 1) and 2).

∎

Let M be an R-n-module and S an non-empty n-submodule of M. Can one establish a connection (as in the binary case) between the n-submodules of M/S and certain n-submodules of M? An answer to this question will be given in the sequel.
We define the map $J_S : \mathcal{S}_{\ell m}(M) \to \mathcal{S}_{\ell m}(M)$ by $J_S(T) = p^{-1}(p(T))$, for any $T \in \mathcal{S}_{\ell m}(M)$, where p is the natural homomorphism. The map J_S is well defined, since T is an n-submodule of M and p is a homomorphism. Note that $J_S(T)$ can be described also as:

$$J_S(T) = \{y \in M \mid \exists s_1, \dots, s_{n-1} \in S : \text{ such that : } y = [x, s_1^{n-1}]_+\}.$$

Remark as well that $p(J_S(T)) = p(T)$ and that $J_S(T)$ is the biggest n-submodule of M with this property. The mapping $J_S(T)$ is a closure operator (we consider the set $\mathcal{S}_{\ell m}(M)$ partially ordered by set-inclusion).

Proposition 2.14 *Let S be a non-empty n-submodule of an R-n-module M. Then the following hold:*
1) If $T \subseteq S$, then $J_S(T) = S$.
2) If $S \subset T$, then $J_S(T) = T$.
3) $\operatorname{Ker} p = J_S(\mathcal{N}_0)$.
4) $\langle S \cup T \rangle = \langle S \cup J_S(T) \rangle$.
5) $S \cap T \neq \emptyset \Leftrightarrow S \subseteq J_S(T) \Leftrightarrow \langle S \cup T \rangle = J_S(T)$.
6) $S \cap T = \emptyset \Leftrightarrow J_S(T) \cap S = \emptyset$.
7) If $\mathcal{N}_0 \subseteq S$, then $J_S(T) = T$ if and only if $S \subseteq T$.

Proof. Statements 1) and 2) are immediate.
3) We have

$$x \in \operatorname{Ker} p \Leftrightarrow p(x) \in \mathcal{N}_{0\,M/S} \Leftrightarrow \exists e \in \mathcal{N}_0 : e \in p(x)$$
$$\exists e \in \mathcal{N}_0, s_1, \ldots, s_{n-1} \in S : x = [e, s_1^{n-1}]_+ \Leftrightarrow x \in J_S(\mathcal{N}_0).$$

4) From $T \subseteq J_S(T)$ it follows that $\langle S \cup T \rangle \subseteq \langle S \cup J_S(T) \rangle$. Conversely, take $x \in \langle S \cup J_S(T) \rangle$; it follows that $\exists x_1, \ldots, x_k \in S, x_{k+1}, \ldots, x_n \in J_S(T)$ such that $x = [x_1^n]_+$. For $i = k+1, \ldots, n$ there exist $t_i \in T, s_{i1}, \ldots, s_{i,n-1} \in S$ such that $x_i = [t_i, s_{i1}^{i,n-1}]_+$. We have now

$$x = [x_1^k, t_{k+1}^n, s_{k+1,1}^{k+1,n-1}, \ldots, s_{n1}^{n,n-1}]_+ \in \langle S \cup T \rangle,$$

which shows that $\langle S \cup J_S(T) \rangle \subseteq \langle S \cup T \rangle$.
5) If $S \cap T \neq \emptyset$, then $\exists e \in \mathcal{N}_0$ with $e \in S \cap T$. Since any $x \in S$ can be written as: $x = [\overset{(n-1)}{e}, x]_+$, it follows that $S \subseteq J_S(T)$. Conversely, if $S \subseteq J_S(T)$, then for any $x \in S$ there exist $t \in T, s_1, \ldots, s_{n-1} \in S$ such that $x = [t, s_1^{n-1}]$; now, since S is an n-submodule, it follows that $t \in S$ and $S \cap T \neq \emptyset$.
If $S \subseteq J_S(T)$, then $\langle S \cup T \rangle \subseteq J_S(T)$. From 4) we have that $J_S(T) \subseteq \langle S \cup T \rangle$.
6) Take $y \in J_S(T)$. Then $\exists x \in T, s_1, \ldots, s_{n-1} \in S$ such that $y = [x, s_1^{n-1}]_+$. If $y \in S$ too, then we obtain $x \in S$, hence $x \in S \cap T$. We have proved that if $S \cap T = \emptyset$, then $J_S(T) \cap S = \emptyset$. The converse follows immediately from $T \subseteq J_S(T)$.
7) Follows from 2) and 5). ∎

We shall say that an n-submodule T of M is *closed with respect to S*, if $T = J_S(T)$.

Theorem 2.15 *The map $T \mapsto p(T)$ is a lattice isomorphism between the set of the n-submodules of M which are closed with respect to S and the set of n-submodules of M/S.*

Proof. Let T_1, T_2 be two closed n-submodules with $p(T_1) = p(T_2)$. Then $p^{-1}(p(T_1)) = p^{-1}(p(T_2))$, i.e. $J_S(T_1) = J_S(T_2)$, or $T_1 = T_2$. Take now an arbitrary $U \in \mathcal{S}_{Rn}(M/S)$ and put $T = p^{-1}(U) \in \mathcal{S}_{Rn}(M)$. We have $p(T) = U$ and $p^{-1}(p(T)) = p^{-1}(U) = T$, which means $J_S(T) = T$.
Finally, if $T_1 \subseteq T_2$ it is known that $p(T_1) \subseteq p(T_2)$. ∎

Corollary 2.16 *If $N_0 \subseteq S$ then the map $T \mapsto T/S$ is a lattice isomorphism between the set of n-submodules of M which contain S and the set of n-submodules of M/S.*

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