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ON A CONTINUOUS DEPENDENCE RESULT FOR EVOLUTION EQUATIONS IN HILBERT SPACES

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Abstract. In this paper we give a new proof for a result concerning the continuous dependence on data for the solutions of a boundary value problem governed by maximal monotone operators in a real Hilbert space.

1. INTRODUCTION

In [1], the author studied the continuous dependence on data for the boundary value problem:

$$u''(t) \in Au(t) \ \text{ a.e. on } \ [0,T] \tag{1.1}$$

$$u(0) = u_0, \ \ u(T) = u_1 \tag{1.2}$$

where A is a maximal monotone operator in the real Hilbert space H, $u_0, u_1 \in \overline{D(A)}$; here $D(A)$ is the domain of A and $T > 0$ is fixed. The existence and the uniqueness of the solution u of this problem were shown by Barbu [4], [5].

Let u_n be the solution of

$$u''(t) \in A^n u_n(t) \ \text{ a.e. on } \ [0,T] \tag{1.3}$$

$$u_n(0) = u_{0n}, \ \ u_n(T) = u_{1n}, \tag{1.4}$$

where A^n are maximal monotone operators in H, $u_{0n}, u_{1n} \in \overline{D(A)}$. The main result of [1] proves that, if $u_{0n} \to u_0$, $u_{1n} \to u_1$ strongly in H and $(A^n)_n$ converges to A in the sense of resolvent (see definition 2.1), then $u_n(t) \to u(t)$, $n \to \infty$, uniformly on $[0,T]$.

A similar result for the first order differential equation

$$u'(t) + Au(t) \ni f(t), \ \text{ a.e. on } \ [0,T], \ \ u(0) = u_0 \tag{1.5}$$

was established by Brezis and Pazy [8], [9].

In [2], the author gave an analogous theorem concerning the problem on half-axis:

$$u''(t) \in Au(t), \text{ a.e. on } [0, \infty) \tag{1.6}$$

$$u(0) = u_0, \ \|u(t)\| \leq C, \ (\forall) t \geq 0. \tag{1.7}$$

The existence for this problem was shown by Barbu [5].

If u_n is the solution of

$$u_n''(t) \in A^n u_n(t), \text{ a.e. on } [0, \infty) \tag{1.8}$$

$$u_n(0) = u_{0n}, \ \sup_{t \geq 0} \|u_n(t)\| = C_n < \infty \tag{1.9}$$

and if $u_{0n} \to u_0$, $(A^n)_n$ converges in the sense of resolvent to A, then $u_n(t) \to u(t)$, $n \to \infty$, uniformly on every compact $[0, L]$ of $[0, \infty)$.

In this paper, we give another proof for the main result of [2].

2. THE RESULT

Let H be a real Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\| \ \|$. Let A, A^n be maximal monotone operators with $0 \in A0$, $0 \in A^n 0$ and $u_0 \in \overline{D(A)}$, $u_{0n} \in \overline{D(A^n)}$..

We define the resolvent of A as $\mathcal{J}_\lambda = (I + \lambda A)^{-1}$ and the Yosida approximation of A as the single-valued operator $A_\lambda = (I - \mathcal{J}_\lambda)/\lambda$, $\lambda > 0$. Denote $A^0 x$ (with $x \in D(A)$) the element of least norm in Ax, i.e.

$$\|A^0 x\| = inf\{\|y\|;\ y \in Ax\}.$$

The theory of maximal monotone operators may be find in [6] or [7].

DEFINITION 2.1. We say that $(A^n)_n$ converges to A in the sense of resolvent if

$$(I + \lambda A^n)^{-1} a \to (I + \lambda A)^{-1} a, \tag{2.1}$$

for every $a \in D(A)$ and $\lambda > 0$.

About different types of convergence of (A^n), the reader may refer to Attouch [3].

The main result of [2] is:

THEOREM 2.1. *Let A, A^n be maximal monotone operators in H and C, C_n a sequence of positive constants. Assume that $(A^n)_n$ converges to A in the sense of resolvent,*

$u_0 \in \overline{D(A)}$, $u_{0n} \in \overline{D(A^n)}$, $u_{0n} \to u_0$ strongly in H, $0 \in D(A) \cap D(A^n)$, $0 \in A0$, $0 \in A^n 0$. If u and u_n are the solutions of the problems (1.6)-(1.7), respectively (1.8)-(1.9), then

$$u_n(t) \to u(t), \text{ as } n \to \infty \tag{2.2}$$

uniformly on every compact $[0, L]$ of $[0, \infty)$.

Remark. Denote by $A_{1/2}$ and $A^n_{1/2}$ the square roots of the operators A and A^n (Barbu [5], [6]) and let $S_{1/2}(t)$, $S^n_{1/2}(t)$, $t \geq 0$, the semigroups of nonlinear contractions generated by $A_{1/2}$, respectively $A^n_{1/2}$. Theorem 2.1 states that, under the above hypotheses,

$$S^n_{1/2}(t)u_{0n} \to S_{1/2}(t)u_0, \text{ as } n \to \infty, \tag{2.3}$$

uniformly on every compact of $[0, \infty)$.

The sequence $(C_n)_n$ is bounded (see Lemma 2.1, [2]).

3. THE NEW PROOF OF THEOREM 2.1

Here we use the techniques of [1]. Let $\lambda > 0$ be given. Set $y_\lambda = (I + \sqrt{\lambda}A)^{-1}u_0$, $y_{n\lambda} = (I + \sqrt{\lambda}A^n)^{-1}u_0$. From (2.1), we get:

$$y_{n\lambda} \to y_\lambda, \text{ as } n \to \infty, (\forall)\lambda > 0 \tag{3.1}$$

$$A^n_\lambda \alpha \to A_\lambda \alpha, \ (\forall)\alpha \in D(A), \ \lambda > 0 \ (\text{as } n \to \infty). \tag{3.2}$$

Here A^n_λ is the Yosida approximation of A^n. Denote by w_λ, v_λ, $w_{n\lambda}$, $v_{n\lambda}$ the solutions of the following problems:

$$w''_\lambda \in Aw_\lambda, \text{ a.e. on } [0, \infty), \ w_\lambda(0) = y_\lambda, \ \|w_\lambda(t)\| \leq C_1, \ (\forall)t \geq 0 \tag{3.3}$$

$$v''_\lambda = A_\lambda v_\lambda, \text{ a.e. on } [0, \infty), \ v_\lambda(0) = y_\lambda, \ \|v_\lambda(t)\| \leq C_2, \ (\forall)t \geq 0 \tag{3.4}$$

$$w''_{n\lambda} \in A^n w_{n\lambda}, \text{ a.e. on } [0, \infty), \ w_{n\lambda}(0) = y_{n\lambda}, \ \|w_{n\lambda}(t)\| \leq C_3, \ (\forall)t \geq 0 \tag{3.5}$$

$$v''_{n\lambda} = A^n_\lambda v_{n\lambda}, \text{ a.e. on } [0, \infty), \ v_{n\lambda}(0) = y_{n\lambda}, \ \|v_{n\lambda}(t)\| \leq C_4, \ (\forall)t \geq 0. \tag{3.6}$$

We have omitted the argument t, for the simplification of the writting. Then, for every $t \in [0, \infty)$, $n \in \mathbb{N}$ and $\lambda > 0$, we have:

$$\|u_n - u\| \leq \|u_n - w_{n\lambda}\| + \|w_{n\lambda} - v_{n\lambda}\| + \|v_{n\lambda} - v_\lambda + \|v_\lambda - w_\lambda\| + \|w_\lambda - u\|. \tag{3.7}$$

To show that

$$\limsup_{n \to \infty} \|u_n - u\| = 0,$$

we need some auxiliary results:

LEMMA 3.1. *If the above hypotheses hold, then, for every $\lambda > 0$, the sequences $(v''_{n\lambda})$ and $(v'_{n\lambda})$ are bounded in $L^2(0,\infty;H)$, with respect to n.*

Proof. We multiply (3.6) by $v''_{n\lambda}$ and we integrate on $[0,\infty)$:

$$\int_0^\infty \|v''_{n\lambda}\|^2 dt \le (A^n_\lambda v_{n\lambda},\ v'_{n\lambda})\ |_0^\infty \le \|(A^n)^0 y_{n\lambda}\|\ \|v'_{n\lambda}(0)\|, \tag{3.8}$$

where $(A^n)^0$ is the minimal section of A^n. We used the fact that $v_{n\lambda} \in H^2(0,\infty;H)$ (see Barbu [5]), so

$$\lim_{t\to\infty}(A^n_\lambda v_{n\lambda}(t),\ \ v'_{n\lambda}(t)) = 0.$$

But, $(u_0 - y_{n\lambda})/\sqrt{\lambda} = [u_0 - (I + \sqrt{\lambda}A^n)^{-1}u_0]/\sqrt{\lambda} = A^n_{\sqrt{\lambda}}(u_0) \in A^n(\mathcal{J}^n_{\sqrt{\lambda}}u_0) = A^n y_{n\lambda}$, so

$$\|(A^n)^0 y_{n\lambda}\| \le \|A^n_{\sqrt{\lambda}}u_0\| = \|u_0 - y_{n\lambda}\|/\sqrt{\lambda}. \tag{3.9}$$

From (3.8) and (3.9), we have

$$\int_0^\infty \|v''_{n\lambda}\|^2 dt \le \frac{1}{\sqrt{\lambda}}\|u_0 - y_{n\lambda}\|\ \|v'_{n\lambda}(0)\|. \tag{3.10}$$

Since $y_{n\lambda} \to y_\lambda$ (strongly in H) as $n \to \infty$, $\|u_0 - y_{n\lambda}\|/\sqrt{\lambda} \le B_\lambda$, which is a positive constant with respect to n.

$$\int_0^\infty \|v''_{n\lambda}\| dt \le B_\lambda \|v'_{n\lambda}(0)\|,\ \ n \in I\!N. \tag{3.10'}$$

Multiplying (3.6) by $v_{n\lambda}$, we obtain

$$(v''_{n\lambda},\ v_{n\lambda}) = (A^n_\lambda v_{n\lambda}, v_{n\lambda}) \ge 0,$$

because A^n_λ is maximal monotone and $A^n_\lambda 0 = 0$. This implies

$$(v'_{n\lambda}, v_{n\lambda})' - \|v'_{n\lambda}\|^2 \ge 0,$$

therefore

$$\int_0^\infty \|v'_{n\lambda}\|^2 dt \le -(v'_{n\lambda}(0), y_{n\lambda}) \le C_\lambda \|v'_{n\lambda}(0)\|,\ \ n \in I\!N, \tag{3.11}$$

where $C_\lambda \ge \|y_{n\lambda}\|$. Let show now the boundedness of $v'_{n\lambda}(0)$ in H (with respect to n):

A multiplication of (3.6) by $A^n_\lambda v_{n\lambda}$, followed by an integration by parts over $[0,\infty)$, leads to

$$\int_0^\infty \|A^n_\lambda v_{n\lambda}\|^2 dt = (v'_{n\lambda}, A^n_\lambda v_{n\lambda})\ |_0^\infty - \int_0^\infty (v'_{n\lambda}, (A^n_\lambda v_{n\lambda})')dt.$$

But it is well known that $(v'_{n\lambda}, (A^n_\lambda v_{n\lambda})')(t) \geq 0$ a.e. on $[0, \infty)$, hence

$$\int_0^\infty \|A^n_\lambda v_{n\lambda}\|^2 dt \leq B_\lambda \|v'_{n\lambda}(0)\|. \tag{3.12}$$

Finally, we multiply (3.6) by $v'_{n\lambda}$, therefore $(1/2)\|v'_{n\lambda}\|^2 \,|_0^\infty = \int_0^\infty (A^n_\lambda v_{n\lambda}, v'_{n\lambda})dt$, or, equivalently,

$$\|v'_{n\lambda}(0)\|^2 \leq 2\left(\int_0^\infty \|A^n_\lambda v_{n\lambda}\|^2 dt\right)^{1/2} \cdot \left(\int_0^\infty \|v'_{n\lambda}\|^2 dt\right)^{1/2}. \tag{3.13}$$

Using (3.12) and (3.11) in (3.13), we obtain

$$\|v'_{n\lambda}(0)\| \leq 2\sqrt{B_\lambda C_\lambda}, \tag{3.14}$$

so, $v'_{n\lambda}(0)$ is bounded in H with respect to n (for $\lambda > 0$ fixed) and, from (3.10)', (3.11), we have the boundedness of $v''_{n\lambda}$ and $v'_{n\lambda}$ in $L^2(0, \infty; H)$.

LEMMA 3.2. *Let A be a maximal monotone operator of H, $0 \in A0$, $u_0 \in D(A)$, $y_\lambda = (I + \sqrt{\lambda}A)^{-1}u_0$, $\lambda > 0$. If w_λ and v_λ are the solutions of (3.5) and (3.3), (3.4), respectively, then w''_λ bounded is in $L^2(0, \infty; H)$, with respect to λ and there is $C > 0$ independent of λ, such that*

$$\|v_\lambda(t) - w_\lambda(t)\| \leq \sqrt{Ct\lambda}. \tag{3.15}$$

Proof. Since $(1/2)(\|v_\lambda - w_\lambda\|^2)'' = (v''_\lambda - w''_\lambda, v_\lambda - w_\lambda) + \|v'_\lambda - w'_\lambda\|^2$, from (3.3) we get

$$\frac{1}{2}(\|v_\lambda - w_\lambda\|^2)'' \geq \lambda(A_\lambda v_\lambda - w''_\lambda, A_\lambda v_\lambda) + \|v'_\lambda - w'_\lambda\|^2,$$

hence

$$\int_0^\infty \|v'_\lambda - w'_\lambda\|^2 dt \leq -\lambda \int_0^\infty (A_\lambda v_\lambda - w''_\lambda, A_\lambda v_\lambda)dt + (v'_\lambda - w'_\lambda, v_\lambda - w_\lambda)\,|_0^\infty.$$

But, $-2(A_\lambda v_\lambda - w''_\lambda, A_\lambda v_\lambda) \leq \|w''_\lambda\|^2 - \|A_\lambda v_\lambda\|^2$, which implies that

$$\int_0^\infty \|v'_\lambda - w'_\lambda\|^2 dt \leq (\lambda/2)\int_0^\infty \|w''_\lambda\|^2 dt. \tag{3.16}$$

We show that w''_λ is bounded in $L^2(0, \infty; H)$, approximating w_λ by the solution $x_{\mu\lambda}$ of the problem

$$x''_{\mu\lambda} = A_\mu x_{\mu\lambda} \text{ a.e. on } [0, \infty), \quad x_{\mu\lambda}(0) = y_\lambda, \quad \sup_{t \geq 0}\|x_{\mu\lambda}(t)\| < \infty. \tag{3.17}$$

Like in the proof of Lemma 3.1, we obtain that

$$\int_0^\infty \|x''_{\mu\lambda}\|^2 dt \leq C, \quad (\forall)\lambda, \mu > 0, \tag{3.18}$$

so, $\int_0^\infty \|w_\lambda''\|^2 dt \le C$. Now

$$\|v_\lambda(t) - w_\lambda(t)\| = \|\int_0^t (v_\lambda' - w_\lambda')ds\| \le \sqrt{t}(\int_0^\infty \|v_\lambda' - w_\lambda'\|^2 ds)^{1/2},$$

and from (3.16), we arrive to (3.15), as claimed.

LEMMA 3.3. *Under the hypotheses of Theorem 2.1, we have*

$$\int_0^\infty \|w_{n\lambda}''\|^2 dt \le 2\|u_0 - y_{n\lambda}\|^{3/2}\|y_{n\lambda}\|^{1/2}/\sqrt[4]{\lambda^3}, \tag{3.19}$$

$$\limsup_{n\to\infty}\|v_{n\lambda}(t) - w_{n\lambda}(t)\| \le \sqrt[8]{\lambda}\cdot\sqrt{Ct},\ \ t \ge 0,\ \ \lambda > 0. \tag{3.20}$$

Proof. As in the proof of Lemma 3.2, relation (3.16), we find here

$$\int_0^\infty \|v_{n\lambda}' - w_{n\lambda}'\|^2 dt \le (\lambda/2)\int_0^\infty \|w_{n\lambda}''\|^2 dt. \tag{3.21}$$

It is enough to prove (3.19). For this, let us approximate $w_{n\lambda}$ by $x_{n\mu\lambda}$, the solution of

$$x_{n\mu\lambda}'' = A_\mu^n x_{n\mu\lambda},\ \text{ a.e. on } [0,\infty),\ \ n \in I\!N \tag{3.22}$$

$$x_{n\mu\lambda}(0) = y_{n\lambda},\ \ sup_{t\ge 0}\|x_{n\mu\lambda}(t)\| < \infty. \tag{3.23}$$

As in the proof of Lemma 3.1 and Lemma 3.2, we obtain

$$\int_0^\infty \|x_{n\mu\lambda}''\|^2 dt \le 2\|u_0 - y_{n\lambda}\|^{3/2}\|y_{n\lambda}\|^{1/2}/\sqrt[4]{\lambda^3}. \tag{3.24}$$

Since $x_{n\mu\lambda}''$ is bounded in $L^2(0,\infty;H)$ with respect to μ, one has $x_{n\mu\lambda}'' \rightarrow w_{n\lambda}''$ in $L^2(0,\infty;H)$ (weak convergence) and then

$$\int_0^\infty \|w_{n\lambda}''\|^2 dt \le \liminf_{\mu\to 0}\int_0^\infty \|x_{n\mu\lambda}''\|^2 dt \le 2\|u_0 - y_{n\lambda}\|^{3/2}\|y_{n\lambda}\|^{1/2}/\sqrt[4]{\lambda^3},$$

which is (3.19). From (3.21), we have

$$\limsup_{n\to\infty}\int_0^\infty \|v_{n\lambda}' - w_{n\lambda}'\|^2 dt \le \sqrt[4]{\lambda}\|u_0 - y_\lambda\|^{3/2}\|y_\lambda\|^{1/2}. \tag{3.25}$$

Since, $v_{n\lambda}(t) - w_{n\lambda}(t) = \int_0^t (v_{n\lambda}'(s) - w_{n\lambda}'(s))ds$, we obtain (3.20).

LEMMA 3.4. *Under the hypotheses of Theorem 2.1, for every $L > 0$ (fixed),*

$$\lim_{n\to\infty}\int_0^L \|v_{n\lambda}' - v_\lambda'\|^2 dt = 0,\ \ (\forall)\lambda > 0. \tag{3.26}$$

Proof. From (3.6) and (3.4), we deduce that $(v_{n\lambda}'' - v_\lambda'', v_{n\lambda} - v_\lambda) = (A_\lambda^n v_{n\lambda} - A_\lambda v_\lambda, v_{n\lambda} - v_\lambda)$, $(\forall)t \ge 0$, or, equivalently,

$$(v_{n\lambda}' - v_\lambda', v_{n\lambda} - v_\lambda)' - \|v_{n\lambda}' - v_\lambda'\|^2 = (A_\lambda^n v_{n\lambda} - A_\lambda^n v_\lambda, v_{n\lambda} - v_\lambda) + (A_\lambda^n v_\lambda - A_\lambda v_\lambda, v_{n\lambda} - v_\lambda).$$

One integrates from 0 to ∞ and one uses the monotonicity of A_λ^n, so

$$\int_0^L \|v'_{n\lambda} - v'_\lambda\|^2 dt \le -\int_0^\infty (A_\lambda^n v_\lambda - A_\lambda v_\lambda, v_{n\lambda} - v_\lambda) dt - (v'_{n\lambda}(0) - v'_\lambda(0), y_{n\lambda} - y_\lambda). \quad (3.27)$$

The relation (3.1) and the boundedness of $(v'_{n\lambda}(0))_n$ in H (see the proof of Lemma 3.1), lead to

$$\lim_{n\to\infty} (v'_{n\lambda}(0) - v'_\lambda(0), y_{n\lambda} - y_\lambda) = 0. \quad (3.28)$$

The sequence $(v_{n\lambda}(t))_n$ is bounded in n, uniformly on every compact $[0, L] \subseteq [0, \infty)$. According to (3.2), we have

$$\lim_{n\to\infty} \int_0^L (A_\lambda^n v_\lambda - A_\lambda v_\lambda, v_{n\lambda} - v_\lambda) dt = 0, \ (\forall)\lambda > 0. \quad (3.29)$$

Now, (3.29), (3.28) and (3.27) imply (3.26), as claimed.

Remark. But, $\|v_{n\lambda}(t) - v_\lambda(t)\| \le \|y_{n\lambda} - y_\lambda\| + \sqrt{L} \cdot \left(\int_0^L \|v'_{n\lambda} - v'_\lambda\|^2 dt\right)^{1/2}$, $t \in [0, L]$, so

$$\lim_{n\to\infty} \|v_{n\lambda}(t) - v_\lambda(t)\| = 0, \ (\forall)\lambda > 0, \quad (3.30)$$

uniformly with respect to $t \in [0, L]$.

Finish of the proof of Theorem 2.1. Subtracting (1.6) and (3.3) and multiplying by $w_\lambda - u$, we find:

$$(w''_\lambda - u'', w_\lambda - u) \ge 0, \ \text{ a.e. on } [0, \infty).$$

This implies that

$$\frac{1}{2}(\|w_\lambda - u\|^2)'' = (w''_\lambda - u'', w_\lambda - u) + \|w'_\lambda - u'\|^2 \ge 0, \ \text{ a.e. on } [0, \infty),$$

so, the function $t \to (1/2)\|w_\lambda(t) - u(t)\|^2$ is convex on $[0, \infty)$ and, since it is bounded, it is nonincreasing on $[0, \infty)$ and consequently

$$\|w_\lambda(t) - u(t)\| \le \|y_\lambda - u_0\|, \ \text{ a.e. } t \in [0, \infty). \quad (3.31)$$

Similarly we have

$$\|w_{n\lambda}(t) - u_n(t)\| \le \|y_{n\lambda} - u_{0n}\|, \ \text{ a.e. on } [0, \infty). \quad (3.32)$$

Passing to the superior limit in (3.7), we find, in accordance with (3.32), (3.31), (3.30), (3.20) and (3.15):

$$\limsup_{n\to\infty} \|u_n(t) - u(t)\| \le 2\|y_\lambda - u_0\| + \sqrt[3]{\lambda}\sqrt{Ct} + \sqrt{Ct\lambda}. \quad (3.33)$$

Now, let $\lambda \downarrow 0$ in (3.33). Then, $u_n(t) \to u(t)$ in H, as $n \to \infty$, uniformly with respect to $t \in [0, L] \subseteq [0, \infty)$. The theorem is proved.

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