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SOME PROPERTIES OF m - FAMILIES

Mircea BALAJ

1 Introduction

Let m be a positive integer. The notion which give the title of our paper was introduced by Klee [7] and it one defined as follows:

A family $\mathcal{A}$ of $m+1$ convex subsets of a vector space E will be called an m - *family* in E if $\cap\mathcal{A} = \emptyset$, but each m members of $\mathcal{A}$ have a common point.

From Helly's theorem [6], it immediately follows that, if E is finite-dimensional then in E there is no m - family for $m > \dim E$. The aim of this work is to point out some properties of the m - families. The main result in [2] will receive a new proof (Theorem 6).

If m is a positive integer by $\langle m \rangle$ we denote the set $\{1, 2, ..., m\}$. For an m - family $\{A_1, A_2, ..., A_{m+1}\}$ we shall introduce the following notations:

$A_i' = \cap \{A_j : j \in \langle m+1 \rangle \setminus \{i\}\}$ for each $i \in \langle m+1 \rangle$;

$C = \text{conv}(\cup \{A_i' : i \in \langle m+1 \rangle\})$ and $B_i = A_i \cap C$ for each $i \in \langle m+1 \rangle$.

2 Basic results

A family $\mathcal{A}$ of sets in vector space E is said to be *in general position* [3] if any m - flat, $m < \min(\text{card } \mathcal{A}, \dim E)$ intersects at most $m+1$ members of $\mathcal{A}$.

THEOREM 1. *If $\{A_1, A_2, ..., A_{m+1}\}$ is an m - family in a vector space E, then the family $\left\{A_1', A_2', ..., A_{m+1}'\right\}$ is in general position.*

Proof. According to the remark made in the introductive section, $m+1 \leq \dim E$. By way of contradiction suppose that there exists an $(m-1)$ - flat L which intersects each set A_i'. Then, denoting by $A_i^* = A_i \cap L$, for each $i \in \langle m+1 \rangle$, we have

$$\cap \left\{ A_j^* : j \in \langle m+1 \rangle \setminus \{i\} \right\} = \cap \{ A_j \cap L : j \in \langle m+1 \rangle \setminus \{i\} \} =$$

$$= (\cap \{ A_j : j \in \langle m+1 \rangle \setminus \{i\} \}) \cap L = A_i' \cap L \neq \emptyset.$$

Therefore $\left\{ A_1^*, A_2^*, ..., A_{m+1}^* \right\}$ is a family of convex sets in an $(m-1)$ - flat having the property that each m sets have a common point. By Helly's theorem is follows that

$$\cap \{ A_i^* : i \in \langle m+1 \rangle \} \neq \emptyset$$

and by $A_i^* \subset A_i$ one deduces $\cap \{ A_i : i \in \langle m+1 \rangle \} \neq \emptyset$, relation which violates the hypothesis. ■

THEOREM 2. *If* $\{A_1, A_2, ..., A_{m+1}\}$ *is an* m *- family in a vector space* E*, then the following statements holds:*

(*a*) $\cap \{ B_j : j \in \langle m+1 \rangle \setminus \{i\} \} = A_i'$ *for each* $i \in \langle m+1 \rangle$;

(*b*) $\cap \{ B_i : i \in \langle m+1 \rangle \} = \emptyset$;

(*c*) *conv*$(A_i' \cap B_i) = C$ *for each* $i \in \langle m+1 \rangle$.

Proof. (*a*) For every $i \in \langle m+1 \rangle$ the inclusion $A_i' \subset C$ implies

$$A_i' = A_i' \cap C = \cap \{ A_j : j \in \langle m+1 \rangle \setminus \{i\} \} \cap C =$$

$$= \cap \{ A_j \cap C : j \in \langle m+1 \rangle \setminus \{i\} \} = \cap \{ B_j : j \in \langle m+1 \rangle \setminus \{i\} \}.$$

The assertion (*b*) one obtains immediately by the inclusion $B_i \subset A_i$ $(i \in \langle m+1 \rangle)$ and by $\cap \{ A_i : i \in \langle m+1 \rangle \} = \emptyset$.

(*c*) Let $x \in C$. The sets A_i', $i \in \langle m+1 \rangle$, being convex x admits the representation (see [9, p.165]) $x = \sum_{i=1}^{m+1} \alpha_i x_i$, where $x_i \in A_i'$, $\alpha_i \geq 0$ $(i \in \langle m+1 \rangle)$ and $\sum_{i=1}^{m+1} \alpha_i = 1$. If $\alpha_1 = 1$, then $x = x_1 \in A_1' \subset \operatorname{conv}(A_1' \cup B_1)$. In contrary case, we can write

$$x = \alpha_1 x_1 + (1 - \alpha_1) \left(\frac{\alpha_2}{1 - \alpha_1} x_2 + ... + \frac{\alpha_{m+1}}{1 - \alpha_1} x_{m+1} \right).$$

By (a) we obtain $x_i \in A'_i \subset B_1$ for each $i \neq 1$, and since the set B_1 is convex we have

$$\frac{\alpha_2}{1-\alpha_1}x_2 + ... + \frac{\alpha_{m+1}}{1-\alpha_1}x_{m+1} \in B_1.$$

Therefore in both cases we have $x \in \text{conv}(A'_1 \cup B_1)$, hence $C \subset \text{conv}(A'_1 \cup B_1)$. The reverse inclusion being obvious we obtain $C = \text{conv}(A'_1 \cup C_1)$. Analogously one establishes that (c) holds for each $i \in \langle m+1 \rangle$. ■

Remark 1. By the previous theorem it follows that if $\{A_1, A_2, ..., A_{m+1}\}$ is an m - family then $\{B_1, B_2, ..., B_{m+1}\}$ is an m - family to.

The proof of Theorem 4 is effected by means of the following known result (see [1], [4]).

LEMMA 3. *Let C be a nonempty convex set in a topological vector space and $\mathcal{A} = \{A_1, A_2, ..., A_n\}$ a family of convex subsets of C such that:*

(i) $\cup\mathcal{A} = C$;

(ii) each $n-1$ members of $\mathcal{A}$ have a common point.

If all members of $\mathcal{A}$ are closed in C, or all are open in C, then $\cap\mathcal{A} \neq \emptyset$.

THEOREM 4. *Let $\{A_1, A_2, ..., A_{m+1}\}$ be an m - family in a topological vector space. If the sets $A_1, A_2, ..., A_{m+1}$ are all closed, or all open, then $\cup\{B_i : i \in \langle m+1 \rangle\}$ is strict included in C.*

Proof. By way of contradiction suppose that $\cup\{B_i : i \in \langle m+1 \rangle\} = C$. This supposition together with the statement (a) in Theorem 2 permit us to apply Lemma 3 to the family $\{B_1, B_2, ..., B_{m+1}\}$. Thus one obtains $\cap\{B_i : i \in \langle m+1 \rangle\} \neq \emptyset$. But this contradicts (b) in Theorem 2 and the proof is complete. ■

THEOREM 5. *Let $\{A_1, A_2, ..., A_{m+1}\}$ be an m - family $(m \geq 2)$ of which members are:*

(a) open convex subsets of a topological vector space

or

(*b*) *closed convex subsets of a locally convex space.*

If $x \in C \setminus (\cup \{B_i : i \in \langle m+1 \rangle\})$, *then for each* $i \in \langle m+1 \rangle$ *there exists a closed hyperplane* H_i *such that:*

(*i*) $x_i \in H_i$;

(*ii*) $H_i \cap A_i = \emptyset$;

(*iii*) $H_i \cap A_j \neq \emptyset$ *for every* $j \in \langle m+1 \rangle \setminus \{i\}$.

Proof. Let $i \in \langle m+1 \rangle$ arbitrarily fixed. Observe that $x \notin A_i$ since in contrary case it would follow $x \in A_i \cap C = B_i$, what would contradict the choice of x.

We shall prove the existence of the hyperplane H_i which satisfies (*i*) and (*ii*), separately for each of the cases (*a*) and (*b*).

(*a*) Since the affine set $\{x\}$ is disjoint from the open convex set A_i, by Mazur's theorem (see [8, p.167]) there exists a closed hyperplane H_i that contains x and is disjoint from A_i.

(*b*) According to a classical separation theorem (see [8, p.171]), the disjoint convex sets $\{x\}$ and A_i, being one compact and the other closed, are strictly separated by a closed hyperplane H. Denoting by H_i the translate of H which contains x, it is clear that A_i and H_i have not common points.

Once determined the hyperplane H_i let us to show that it satisfies (*iii*). Let S_1 be the open halfspaces bordered by H_i, which contains the set A_i and let S_2 be the opposed open halfspace. From the construction of the sets A'_j it follows that for $j \neq i$ $A'_j \subset A_i$, hence $A'_j \subset S_1$. Remark now that the set A'_i can not be included in S_1. Indeed, supposing $A'_i \subset S_1$ it would follow

$$C \subset \operatorname{conv}\left(\cup \left\{A'_j : j \in \langle m+1 \rangle\right\}\right) \subset S_1,$$

what contradicts $x \in H_i \cap C$. Therefore $A'_i \cap S_1 = \emptyset$ and by $A'_i \subset A_j$ for $j \neq i$ we deduce that $A_j \cap \bar{S}_2 \neq \emptyset$ for each $j \neq i$. Every set A_j intersects the set A_1, hence it has common points with S_1. Thus, by the convexity of the sets A_j, it follows (*iii*). ■

Remark 2. Theorem 5 is not true in the case $m = 1$. For motivate this, it is sufficient to consider A_1 and A_2 as being two opposed open halfspaces (case (a)), respectively two opposed sides of a square in $\mathbf{R}^2$ and x the centre of the square (case (b)).

We close our paper giving a new proof for the main result (Theorem 3) in [2].

THEOREM 6. *Let* $\{A_1, A_2, ..., A_{m+1}\}$ *be an* m *- family of*

(a) *open convex subsets of a topological vector space* E

or

(b) *closed convex subsets of a locally convex space* E, *at least one being compact.*

Then there is a closed flat L *in* E, *of deficiency* m, *such that:*

(i) $L \cap (\cup \{A_i : i \in \langle m+1 \rangle\}) = \emptyset$;

(ii) $L \cap$ *conv*$(\cup \{A_i : i \in \langle m+1 \rangle\}) \neq \emptyset$.

Proof. The proof is by induction on m. In the case $m = 1$ the required flat is a hyperplane. For two disjoint convex sets A_1, A_2 both open in a topological vector or both closed and at least one compact in a locally convex space, by two standard separation theorems [5, pp.32 and 64] there is a closed hyperplane that strictly separates A_1 from A_2.

Assume inductively that the statement is true for $m-1$ ($m \geq 2$) and let $\{A_1, A_2, ..., A_{m+1}\}$ be an m - family as described in the theorem. In the case (b) we suppose, without loss of generality, that at least one of the sets $A_1, A_2, ..., A_m$ is compact. Fix any x in the set $C \setminus (\cup \{B_i : i \in \langle m+1 \rangle\})$. By Theorem 5 there is a closed hyperplane H which contains x, intersects each of the sets A_i ($i \in \langle m \rangle$) and does not intersect A_{m+1}. Denote by S_1 and S_2 the two open halfspaces bordered by H, supposing $A_{m+1} \subset S_1$.

From $x \in C = \mathrm{conv}(A_{m+1} \cup B_{m+1})$ (see Theorem 2, (c)) and $x \notin B_{m+1}$ it follows that there exist two points $y \in A'_{m+1}$ and $z \in B_{m+1}$ such that x belongs to the open line segment joining y, z. Since $z \in B_{m+1} \subset A_{m+1} \subset S_1$, it follows that $y \in S_2$, hence

$$\cap \{A_i : i \in \langle m \rangle\} \cap S_2 \neq \emptyset. \tag{1}$$

Define $A_i^* = A_i \cap H$ for each $i \in \langle m \rangle$ and we intend to prove that $\{A_1^*, A_2^*, ..., A_m^*\}$ is an $(m-1)$ - family. First we shall show that any $m-1$ sets (for instance $A_1^*, A_2^*, ..., A_{m-1}^*$) have a common point.
By (1) we have $\cap \{A_i : i \in \langle m-1 \rangle\} \cap S_2 \neq \emptyset$. Also

$$\cap \{A_i : i \in \langle m-1 \rangle\} \cap S_1 \supset \cap \{A_i : i \in \langle m-1 \rangle\} \cap A_{m+1} \neq \emptyset.$$

So, the convexity of the set $\cap \{A_i : i \in \langle m-1 \rangle\}$ implies

$$\cap \{A_i^* : i \in \langle m-1 \rangle\} = \cap \{A_i \cap H : i \in \langle m-1 \rangle\} = \cap \{A_i : i \in \langle m-1 \rangle\} \cap H \neq \emptyset.$$

Since $x \in \operatorname{conv}(\cup \{A_i' : i \in \langle m+1 \rangle\})$, there are $x_i \in A_i'$ $(i \in \langle m+1 \rangle)$ such that $x \in \operatorname{conv}\{x_1, x_2, ..., x_{m+1}\}$ (see [9, p.165]). The family $\left\{A_1', A_2', ..., A_{m+1}'\right\}$ being in general position (see Theorem 1), the convex hull of the points x_i $(i \in \langle m+1 \rangle)$ is an m - dimensional simplex which will be denote by Δ. If F_i is the face of Δ which does not contain the vertex x_i $(i \in \langle m+1 \rangle)$ then we have $x_k \in A_k' \subset B_i$ for $k \neq i$ and by the convexity of the sets B_i it follows $F_i \subset B_i$. Since x was chosen from $C \setminus (\cup \{B_i : i \in \langle m+1 \rangle\})$ we deduce that x belongs to the relative interior of simplex Δ.

We claim that $\cap \{A_i^* : i \in \langle m \rangle\} = \emptyset$. Let us suppose that there is a point $y \in \cap \{A_i^* : i \in \langle m \rangle\} = \cap \{A_i : i \in \langle m \rangle\} \cap H$. Then if d is the line through x and y, it will intersect the relative boundary of Δ in two points. Let u be that point of intersection for which x belongs to the open segment $]u, y[$. By $x, y \in H$ it follows $d \subset H$ and since $F_{m+1} \subset B_{m+1} \subset S_1$, the point u can not belong to the face F_{m+1}. Let $j \in \langle m \rangle$ such that $u \in F_j$, hence $u \in B_j$.
As $\cap \{A_i : i \in \langle m \rangle\} = A_{m+1}' = \cap \{B_i : i \in \langle m \rangle\}$, it follows $y \in B_j$. Since $u \in B_j$ and $x \in]u, y[$ one obtains $x \in B_j$, which contradicts $x \in C \setminus (\cup \{B_i : i \in \langle m+1 \rangle\})$. Therefore $\cap \{A_i^* : i \in \langle m \rangle\} = \emptyset$, hence $\{A_1^*, A_2^*, ..., A_m^*\}$ is an $(m-1)$ - family. Obviously the sets A_i^* are open in the case (a), respectively closed and at least one compact in the case (b). The inductive assumption is now applicable to the $(m-1)$ - family $\{A_1^*, A_2^*, ..., A_m^*\}$ in H. Thus there exists a closed flat L of deficiency $m-1$ in H, hence of deficiency m in E such that

$$L \cap (\cup \{A_i^* : i \in \langle m \rangle\}) = \emptyset \text{ and } L \cap \operatorname{conv}(\cup \{A_i^* : i \in \langle m \rangle\}) \neq \emptyset.$$

Since $A_{m+1} \subset S_1$, $L \subset H$ and $A_i^* = A_i \cap H$ it follows that

$$L \cap (\cup \{A_i : i \in \langle m+1 \rangle\}) = \emptyset.$$

Also

$$L \cap \operatorname{conv}(\cup \{A_i : i \in \langle m+1 \rangle\}) \supset L \cap \operatorname{conv}(\cup \{A_i : i \in \langle m \rangle\}) \supset$$

$$\supset L \cap \operatorname{conv}(\cup \{A_i^* : i \in \langle m \rangle\}) \neq \emptyset$$

and the proof is complete. ■

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