Bul. Stiinţ. Univ. Baia Mare, Ser. B,
Matematică-Informatică, Vol. XIV(1998), Nr. 2, 139-142

Sasaki Metric and the Phase Space

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Abstract: In this paper we are interested in the *Riemannian geometry* of the phase space for a conservative dynamical system. An adapted metric (*Sasaki metric*) in the phase space allow us to draw conclusions about the intrinsic properties of the space and about the geodesics of this space, which are here the trajectories of the dynamical system.

1. Sasaki metric

Let **M** be an n-dimensional differential manifold and let $q^i{}_{(i=1,\dots,n)}$ be the local co-ordinates of a point $Q \in$ **M**. In what follows we shall use the summation convention, if not stated otherwise.

Let us consider a metric on **M**, given by:

(1) $$ds^2 = g_{ij}(Q)\, dq^i dq^j$$

(**M**, ds^2) is an n-dimensional Riemannian space (see [8], [14]); we denote this space by $\mathbf{M}^n$.

Using the above ds^2, on $T(\mathbf{M}^n)$ - the tangent bundle of $\mathbf{M}^n$, we introduce the *Sasaki metric* (see [12], [13]) given by:

(2) $$d\sigma^2 = g_{ij}(Q)\, dq^i dq^j + g_{ij}(Q)\, \nabla v^i\, \nabla v^j$$

where $$\nabla v^i = dv^i + \Gamma_l{}^i{}_m\, v^l dq^m$$

and $\Gamma_l{}^i{}_m$ are Christoffel's symbols of the second kind.

In order to find the components of the fundamental metric tensor of $T(\mathbf{M}^n)$ we write $d\sigma^2$ as:

(3) $$d\sigma^2 = G_{\alpha\beta}\, dq^\alpha dq^\beta \quad (\alpha,\beta = 1, 2, 3, \dots, 2n)$$

where $$q^{n+i} = v^i, \quad i = 1, 2, 3, \dots, n,$$

$G_{\alpha\beta}$ are to be determined, being the covariant components of the metric tensor, and

(4) $$G_{\alpha\gamma} G^{\gamma\beta} = \delta_\alpha^\beta = \begin{cases} 1 & \text{pour } \alpha = \beta \\ 0 & \text{pour } \alpha \neq \beta \end{cases}$$

From (2) and (3), using the condition (4), we get :

$$
(5) \qquad \begin{aligned} G_{jk} &= g_{jk} + g_{\beta\gamma} \Gamma_{\mu\ j}^{\ \beta} \Gamma_{\nu\ k}^{\ \gamma} v^{\mu} v^{\nu} \\ G_{j\ (n+k)} &= \Gamma_{\lambda k j}\, v^{\lambda} \\ G_{(n+j)\ (n+k)} &= g_{jk} \end{aligned} \qquad j, k = 1,2,3, \ldots, n.
$$

The $G_{\alpha\beta}$ determined above define the metric of the Riemannian space $(\mathrm{T}(\mathrm{M}^n), d\sigma^2)$ (see [12], [13]).

In this paper we are interested to use some particular and very important properties of the *Sasaki metric* on $\mathrm{T}(\mathrm{M}^n)$, summarised bellow:

In the tangent bundle $\mathrm{T}(\mathrm{M}^n)$ *of any Riemannian manifold* M^n, *every trajectory of the geodesic flow is a lift of a geodesic of* M^n *and every lift of a geodesic of a Riemannian manifold* M^n *is a geodesic of the tangent bundle* $\mathrm{T}(\mathrm{M}^n)$ (see the proof in [12]).

2. Maupertuis metric for a conservative dynamical system

From Jacobi's form of Maupertuis' Least Action Principle, we know (see [1], [3]) that for any fixed value of the constant of energy, **h**, the trajectories of a conservative dynamical system are the geodesics of the configuration space, provided with the *Riemannian Maupertuis metric*:

$$
(6) \qquad ds^2 = 2\,(\mathrm{U}+\mathbf{h})\, ds_0^{\,2},
$$

where $ds_0^{\,2} = 2\,\mathrm{T}\, dt^2$, U is the force function and T is the kinetic energy.

We used for *Maupertuis metric* the same notation as for the metric (1) because we take here M^n to be the configuration space and $\mathrm{T}(\mathrm{M}^n)$ to be the phase space of our conservative dynamical system.

So, what we propose is to endow the configuration space with Maupertuis metric (6) and to use this metric instead of (1), in order to obtain Sasaki metric (5) for the phase space. This way, using the properties of a space endowed with Sasaki metric, the trajectories of the phase space will be the lift of the real trajectories (in the configuration space) of the system. To get more information about these trajectories and about the space itself we may compute, for instance, the Riemannian curvatures (or sectional curvatures) using:

$$
(7) \qquad K_{12} = \frac{\mathrm{R}_{ijkl}\, v_1^i v_2^j v_1^k v_2^l}{\left(g_{jl}\, g_{ik} - g_{jk}\, g_{il}\right) v_1^i v_2^j v_1^k v_2^l},
$$

where $(v_1^{\,i})$ and $(v_2^{\,i})$ are the components of two linear independent vectors of $\mathrm{T}(\mathrm{M}^n)$, R_{ijkl} is the Riemann-Christoffel tensor and g_{ij} are the components of the metric of the space. A discussion on the sign of the riemannian curvatures leads to draw conclusions concerning the stability of trajectories or of some particular configurations.

3. Sasaki metric and the simple pendulum

In order to illustrate the theoretical method proposed and described above, let us consider the classical problem of the simple pendulum. We know that the configuration space is a one-dimensional space and the kinetic energy and the force function are given by:

$$\mathrm{T} = \frac{1}{2} m l^2 \dot{\varphi}^2 \tag{8}$$

$$\mathrm{U} = m\, l\, g \cos(\varphi), \tag{9}$$

where m is the mass of the pendulum, l the length of the wire, g the gravity acceleration and φ the angle between the wire l and the descending direction.

The Maupertuis metric and the corresponding Christoffel symbols of the second kind are:

$$ds_0^2 = \frac{ml^2[mgl\cos(\varphi) + \mathbf{h}]}{2} d\varphi^2 \tag{10}$$

$$\Gamma_{11}^{1} = -\frac{mgl\sin(\varphi)}{2[mgl\cos(\varphi) + \mathbf{h}]} \tag{11}$$

The Riemannian curvatures of the configuration space are all zero.

In the phase space, the components of the Sasaki metric are given in the following table:

$$\begin{bmatrix} \dfrac{ml^2\left\{m^2l^2g^2[p_\varphi^2\sin^2(\varphi) + 4\cos^2(\varphi)] + 4\mathbf{h}[2mgl\cos(\varphi) + \mathbf{h}]\right\}}{8[mgl\cos(\varphi) + \mathbf{h}]} & -\dfrac{m^2l^3g\, p_\varphi \sin(\varphi)}{4} \\ -\dfrac{m^2l^3g\, p_\varphi \sin(\varphi)}{4} & \dfrac{ml^2[mgl\cos(\varphi) + \mathbf{h}]}{2} \end{bmatrix}$$

where $p_\varphi = m l^2 \dot{\varphi}$ is Poisson's variable corresponding to φ.

The non- zero Christoffel symbols of the second kind are:

$$\Gamma_{11}^{1} = -\frac{mgl\sin(\varphi)}{2[mgl\cos(\varphi) + \mathbf{h}]}$$

$$\Gamma_{11}^{2} = -\frac{mglp_\varphi[mgl + \mathbf{h}\cos(\varphi)]}{2[mgl\cos(\varphi) + \mathbf{h}]^2} \tag{12}$$

$$\Gamma_{12}^{2} = -\frac{mgl\sin(\varphi)}{2[mgl\cos(\varphi) + \mathbf{h}]}$$

We used the above expressions to compute the Riemann-Christoffel tensor and the Riemannian curvatures for this space and they all equal zero. Thus, the phase space of the pendulum is flat. We already knew that (using other techniques), but the result proves the utility of the method proposed in this paper. It can be used for qualitative studies in the phase space of any other conservative dynamical system.

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Received 13.05.1998