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A RICCATI TYPE SYSTEM ATTACHED TO A BOUNDARY CONTROL PROBLEM FOR LAPLACE EQUATION

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A boundary control problem with quadratic cost functional for Laplace equation with boundary conditions given by the solution of a differential equation involving control is considered. The form of the feedback for the optimal control and a Riccati type system, which gives a characterization for the kernels that intervient in the feedback form, are derived.

Let $t_0 < T$ be fixed, with $T < \infty$ and $\gamma_i(\cdot):[t_0,T] \to \mathrm{R}$, $i=\overline{1,2}$ be such that $\gamma_1(\cdot) < \gamma_2(\cdot)$ and there exist their first derivative. We suppose that $\dot{\gamma}_i(\cdot)$, $i=\overline{1,2}$ are integrable functions on $[t_0,T]$. $\varphi_1(\cdot) \in \mathcal{C}^2([\gamma_1(t_0),\gamma_2(t_0)];\ \mathbf{R})$, $\varphi_2(\cdot) \in \mathcal{C}^2([\gamma_1(T),\gamma_2(T)];\ \mathbf{R})$. We set $D = \{(t,x); t \in [t_0,T], x \in [\gamma_1(t),\gamma_2(t)]\}$ and suppose that for the domain D and for Laplace operator there exists the Green's function. The controls are, in general, denoted by u, w, ... ; they are taken to be in the space $\mathcal{U} = L^2([t_0,T];\ \mathbf{R}^k)$, $k \geq 1$. $A(\cdot)$, $B(\cdot)$ are matrices of continuous functions of $m \times m$ type and of $m \times k$ type respectively; C is a constant matrix of $2 \times m$ type.

Consider the following boundary value problem, defined on D:

$$(1) \quad \Delta y(t,x) = 0, \ \left(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2}\right)$$

along with boundary conditions,

$$(2)\qquad \begin{cases} y(t_0,x)=\varphi_1(x),\ x\in[\gamma_1(t_0),\gamma_2(t_0)] \\ y(T,x)=\varphi_2(x),\ x\in[\gamma_1(T),\gamma_2(T)] \end{cases}$$

$$(3)\qquad \begin{cases} y(t,\gamma_1(t))=v_1(t) \\ y(t,\gamma_2(t))=v_2(t) \end{cases}$$

$$(4)\qquad v(t)=Cz(t),\ (\forall)t\in[t_0,T],\ v(t)=\begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$$

$$(5)\qquad \begin{cases} \dfrac{dz}{dt}=A(t)z+B(t)u(t),\ u\in\mathcal{U} \\ z(t_0)=z_0,\ \ (z_0\in\mathbb{R}^m,\ \text{is given}) \end{cases}$$

For every $u\in\mathcal{U}$ the problem (5) has a unique solution, let us denote it by $z_u(t)$. The solutions $z_u(\cdot)$ is an absolute continuous function. According to (4) the boundary data $v(\cdot)$ in the problem (1), (2), (3) are as smooth as the solution $z_u(\cdot)$. With the hypothesis on $\varphi_i(\cdot)$, $i=\overline{1,2}$ the problem (1), (2), (3) has a unique solution; let us denote it by $y_u(t,x)$. The function $y_u(\cdot,\cdot)$ is the classical solution of the Laplace equation related to the domain D and it depends continuously on the data $\varphi_i(\cdot)$, $i=\overline{1,2}$ and on the boundary data $v(\cdot)=\begin{pmatrix} v_1(\cdot) \\ v_2(\cdot) \end{pmatrix}$.

For every $u\in\mathcal{U}$, fixed, the solution of the problem (1) - (5) consists of the couple $(y_u(t,x),z_u(t))$.

Let $\Phi(t,s)$ be the fundamental matrix of the solutions of the problem (5) and $G(M,P)$, ($M=(t,x)$, $P=(\sigma,\zeta)$) the Green's function corresponding to the Laplace operator and to the domain D.

The solution of the problem (1) - (5) has the representation

$$(6)\qquad \begin{cases} z_u(t)=\Phi(t,t_0)z_0+\int_{t_0}^{t}\Phi(t,s)B(s)u(s)ds \\ v(t)=Cz(t) \\ y_u(t,x)=-\int_{\gamma_1(t_0)}^{\gamma_2(t_0)}\varphi_1(\zeta)\dfrac{\partial G}{\partial\sigma}(t,x;t_0,\zeta)d\zeta+\int_{\gamma_1(T)}^{\gamma_2(T)}\varphi_2(\zeta)\dfrac{\partial G}{\partial\sigma}(t,x;T,\zeta)d\zeta+ \\ \qquad +\int_{t_0}^{T}v_1(\sigma)\left[-\dfrac{\partial G}{\partial\zeta}(t,x;\sigma,\gamma_1(\sigma))+\dfrac{\partial G}{\partial\sigma}(t,x;\sigma,\gamma_1(\sigma))\dot\gamma_1(\sigma)\right]d\sigma- \\ \qquad -\int_{t_0}^{T}v_2(\sigma)\left[-\dfrac{\partial G}{\partial\zeta}(t,x;\sigma,\gamma_2(\sigma))+\dfrac{\partial G}{\partial\sigma}(t,x;\sigma,\gamma_2(\sigma))\dot\gamma_2(\sigma)\right]d\sigma. \end{cases}$$

We attach to the problem (1) – (5) the following quadratic cost functional

$$J(u)=\int_{t_0}^{T}\int_{\gamma_1(t)}^{\gamma_2(t)}\int_{\gamma_1(t)}^{\gamma_2(t)}K(t;x,\zeta)y_u(t,x)y_u(t,\zeta)\,\mathrm{d}x\,\mathrm{d}\zeta\,\mathrm{d}t+ \tag{7}$$
$$+\int_{\gamma_1(T)}^{\gamma_2(T)}\int_{\gamma_1(T)}^{\gamma_2(T)}G_1(x,\zeta)\frac{\partial y_u}{\partial t}(T,x)\frac{\partial y_u}{\partial t}(T,\zeta)\,\mathrm{d}x\,\mathrm{d}\zeta+$$
$$+\langle G_0z_u(T),z_u(T)\rangle+\int_{t_0}^{T}\langle G_2(t)z_u(t),z_u(t)\rangle\,\mathrm{d}t+\int_{t_0}^{T}\langle H(t)u(t),u(t)\rangle\,\mathrm{d}t,$$

where $K(\cdot\,;\cdot\,,\cdot)$, $H(\cdot)$, G_0, $G_1(\cdot\,,\cdot)$, $G_2(\cdot)$ are as in [1].

The optimal control problem consists in finding of a function $\tilde{u}\in\mathcal{U}$ such that,

$$J(\tilde{u})=\inf_{u\in\mathcal{U}}J(u). \tag{8}$$

The existence and uniqueness of the optimal control are obtained as in [1]. Also we get a necessary and sufficient condition of optimality, namely a condition of (8) type, see[1]. See also [2].

Let us denote by $p(t,x)$, $\psi(t)$ the adjoint state, defined as the solution of the following boundary value problem,

$$\Delta p(t,x)=\int_{\gamma_1(t)}^{\gamma_2(t)}K(t;x,\zeta)y_{\tilde{u}}(t,\zeta)\,\mathrm{d}\zeta \tag{1*}$$

$$\begin{cases}p(t_0,x)=0\\ p(T,x)=-\int_{\gamma_1(T)}^{\gamma_2(T)}G_1(x,\zeta)y_{\tilde{u}}(T,\zeta)\,\mathrm{d}\zeta\end{cases} \tag{2*}$$

$$\begin{cases}p(t,\gamma_1(t))=0\\ p(t,\gamma_2(t))=0\end{cases} \tag{3*}$$

$$\begin{cases}\dfrac{\mathrm{d}\psi}{\mathrm{d}t}=-A^*(t)\psi+G_2(t)z_{\tilde{u}}(t)+C^*w(t)\\ \psi(T)=-G_0z_{\tilde{u}}(T),\end{cases} \tag{5*}$$

where

$$w(t)=\begin{pmatrix}\dfrac{\partial p}{\partial t}(t,\gamma_1(t))\dot{\gamma}_1(t)-\dfrac{\partial p}{\partial x}(t,\gamma_1(t))\\ -\dfrac{\partial p}{\partial t}(t,\gamma_2(t))\dot{\gamma}_2(t)+\dfrac{\partial p}{\partial x}(t,\gamma_2(t))\end{pmatrix}.$$

Using the adjoint state one gets a characterization of the optimal control, given by,

PROPOSITION 1. *The optimal control $\tilde{u}(\cdot)$ admits the representation,*

$$\tilde{u}(t)=H^{-1}(t)B^*(t)\psi(t). \tag{9}$$

For proof see [2].

We consider the following boundary value problem (attached to the Hamiltonian type system):

$$
(10)\quad \begin{cases}
\Delta y(t,x)=0 & \Delta p(t,x)=\int_{\gamma_1(t)}^{\gamma_2(t)} K(t;x,\zeta)\,y(t,\zeta)\,d\zeta \\
y(t,\gamma_1(t))=v_1(t) & p(t,\gamma_1(t))=0 \\
y(t,\gamma_2(t))=v_2(t) & p(t,\gamma_2(t))=0 \\
v(t)=Cz(t) & \\
\dfrac{dz}{dt}=A(t)z+B(t)H^{-1}(t)B^{*}(t)\psi(t) & \dfrac{d\psi}{dt}=-A^{*}(t)\psi+G_2(t)z(t)+C^{*}w(t) \\
 & (\text{with } w(\cdot) \text{ given in } (5^{*})) \\
y(t_0,x)=\varphi_1(x),\ y(T,x)=\varphi_2(x) & p(t_0,x)=0,\quad p(T,x)=-\int_{\gamma_1(T)}^{\gamma_2(T)} G_1(x,\zeta)y(T,\zeta)\,d\zeta \\
z(t_0)=z_0 & \psi(T)=-G_0\,z(T)
\end{cases}
$$

THEOREM 1.

(i) *There exists a unique solution of the boundary value problem* (10). *This is the quadruple* $(y,z;p,\psi)$ *corresponding to the optimal control* $\tilde{u}(\cdot)$ *related to the problem* (1) - (5), (7) – (8).

(ii) *The problem* (10) *can be reduced to a Fredholm integral equation for* $\psi(\cdot)$.

The proaf is similar to that given in [2]. Using the ideas from [2] one gets

$$
(11)\qquad \psi(t)=\lambda(t)+\int_{t_0}^{T} N(t,s)\psi(s)\,ds,
$$

where $\lambda(\cdot)$ depends only on the data $(\varphi_1(\cdot),\varphi_2(\cdot),z_0)$ of the problem (10).

THEOREM 2. *The Fredholm integral equation* (11) *admits a unique solution of the form*

$$
(12)\qquad \psi(t)=\lambda(t)+\int_{t_0}^{T} Q(t,s)\lambda(s)\,ds.
$$

For proof we follow the same technics as in [2] or [3].

THEOREM 3. *The optimal control admits the representation under the linear feedback form*

$$
(13)\qquad \tilde{u}(t)=\int_{\gamma_1(t)}^{\gamma_2(t)} N_1(t,x)y(t,x)\,dx+N_2(t)z(t)+N_3(t),
$$

where $(y(\cdot,\cdot),z(\cdot))$ *are the components y, z of the solution* $(y(\cdot,\cdot),z(\cdot);p(\cdot,\cdot),\psi(\cdot))$ *for the problem* (10).

Proof. We use, see [2], the dynamic programming method. This way we get

$$(14)\quad \begin{cases} \psi(t) = \int_{\gamma_1(t)}^{\gamma_2(t)} M_2(t,\alpha)\, y(t,\alpha)\,\mathrm{d}\alpha + M_4(t)\,z(t) + M_6(t) \\ p(t,x) = \int_{\gamma_1(t)}^{\gamma_2(t)} M_1(t;x,\zeta)\, y(t,\zeta)\,\mathrm{d}\zeta + M_3(t,x)\,z(t) + M_5(t,x) \end{cases}$$

The proof is similar to that from [2] for P 5.1.

According to Th. 3 were given representation formulas for the components (p,ψ) in liniar feedback form. Futher we give a characterization of the kernels involved by the representations (14).

THEOREM 4. *Let* $(y(\cdot,\cdot), z(\cdot))$ *be a solution of the boundary value problem*

$$(15)\quad \begin{cases} \Delta y(t,x) = 0 \\ y(t,\gamma_1(t)) = v_1(t),\ y(t,\gamma_2(t)) = v_2(t) \\ v(t) = Cz(t) \\ \dfrac{\mathrm{d}z}{\mathrm{d}t} = A(t)z + B(t)H^{-1}(t)B^*(t)\left[\int_{\gamma_1(t)}^{\gamma_2(t)} M_2(t,x)\,y(t,x)\mathrm{d}x + M_4(t)z(t) + M_6(t)\right] \\ y(t_0,x) = \varphi(x), \qquad z(t_0) = z_0 \end{cases}$$

Let $(p(\cdot,\cdot), \psi(\cdot))$ *be given by* (14). *Then, for* $(M_1, \dots, M_6)$ *the solution of a certain boundary value problem attached to a certain Riccati type system,* $(y(\cdot,\cdot), z(\cdot))$; $(p(\cdot,\cdot), \psi(\cdot))$ *is a solution of the problem* (10) *and* (9) *together with* (14) *give the optimal control under the feedback form for the problem* (1) – (5), (7) – (8).

REMARK. The procedure used for getting the boundary value problem for $(M_1, \dots, M_6)$ is somehow analogous to that from [2]. We point out that for the kernels M_1, M_2, M_3, M_5 the system consists in partial differential equations, while for the kernel M_4 one gets a Riccati operatorial equation and for the kernel M_6 a differential equation. Because of the Riccati equation for the kernel M_4 we say that the system for $(M_1, \dots, M_6)$ is a Riccati type system.

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