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THE FOURIER-BOHR SERIES OF A PSEUDO ALMOST PERIODIC MEASURE

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Abstract

We define the Fourier-Bohr coefficients of a pseudo almost periodic measure and we establish some of their properties. We also define the Fourier-Bohr series of a pseudo almost periodic measure and we write the corresponding series for the convolution of a bounded measure and a pseudo almost periodic measure.

Finally, we study two convergent Fourier-Bohr series, and we establish a property of their sums.

1. Introduction

In [2] L. Argabright and J. Lamadrid introduced and studied the Fourier-Bohr series for an almost periodic measure. In this paper we generalize this concept and define the Fourier-Bohr coefficients for a pseudo almost periodic measure and the Fourier-Bohr series for such a measure.

Next we establish some properties of the coefficients and we give a necessary and sufficient condition so that a pseudo almost periodic measure be in a certain space of measures. This condition is formulated in terms of the Fourier-Bohr coefficients.

In [6] we proved that the convolution of a bounded measure and a pseudo almost periodic measure is a pseudo almost periodic measure. Now we calculate the Fourier-Bohr coefficients for this convolution and then we write the Fourier-Bohr series.

Finally, we study two convergent Fourier-Bohr series, and we establish a property of their sums.

2. Terminology and notations

Consider a Hausdorff σ-compact, locally compact Abelian group G with the unit element e and let λ be the Haar measure on G.

Let us denote by $\mathcal{C}(G)$ the set of all bounded continuous complex-valued functions on G. The set $\mathcal{C}(G)$ is a Banach space endowed with the supremum norm. Throughout this paper, $\|\cdot\|$ denotes the supremum norm on $\mathcal{C}(G)$. For $f \in \mathcal{C}(G)$ and $a \in G$, the translate of f by a is the function $f_a(x) = f(xa)$ for all $x \in G$. Denote by $K(G)$ the vector space of all continuous complex-valued functions on G, having a compact support.

For each compact subset A of G, we denote by $K(G, A)$ the vector subspace of $K(G)$ consisting of those functions which vanish outside A; for a compact subset A of G, the vector space $K(G, A)$ is a Banach space under the supremum norm. We regard the space $K(G)$ as a topological vector space with the inductive limit topology defined on it by the spaces $K(G, A)$, A being a compact set in G. Let us denote by $m(G)$ the dual of the topological vector space $K(G)$. We use $m_F(G)$ to denote the subspace of $m(G)$ consisting of all bounded measures, that is, all linear functionals which are continuous with respect to the supremum norm on $K(G)$.

The action of a measure $\mu \in m(G)$ on a function $f \in K(G)$ will be denoted either $\mu(f)$ or $\int_G f(x)d\mu(x)$.

Corresponding to a measure $\mu \in m(G)$, one defines the variation measure $|\mu| \in m(G)$ by $|\mu|(f) = \sup\{|\mu(g)| : g \in K(G), |g| \leq f\}$ for all $f \in K(G)$, $f \geq 0$.

Definition 2.1. [7] *The convolution $f * \mu$ of a Borel function f on G with a measure* $\mu \in m(G)$ is the function given by the convolution formula

$$f * \mu(x) = \int_G f(xy^{-1})d\mu(y), \quad x \in G$$

provided that

$$\int_G |f(xy^{-1})| \, d|\mu|(y) < \infty \text{ for all } x \in G.$$

Definition 2.2. [1] We say that two measures $\mu, \nu \in m(G)$ are *convolable* if, for every $f \in K(G)$, the function $(x, y) \in G \times G \to f(xy) \in C$ is integrable over $G \times G$ with respect to the product measure $|\mu| \times |\nu|$.

In this case, the measure $\mu * \nu$ is defined by

$$\mu * \nu(f) = \int_G\int_G f(xy)d\mu(x)d\nu(y) = \int_G\int_G f(xy)d\nu(x)d\mu(y)$$

for all $f \in K(G)$.

Definition 2.3. [2] We say that a measure $\mu \in m(G)$ is *translation-bounded* if for every compact set $A \subseteq G$

$$\sup_{x \in G} \mid \mu \mid (xA) < \infty.$$

The vector space of translation-bounded measures will be denoted by $m_B(G)$.

Remark 2.1. [2] We identify an arbitrary measure $\mu \in m_B(G)$ with an element of the space $[\mathcal{C}_U(G)]^{K(G)}$ by means of the following equation which signifies the identification:

$$\mu = \{f * \mu\}_{f \in K(G)}.$$

From this identification we have the inclusion

$$m_B(G) \subset [\mathcal{C}_U(G)]^{K(G)}.$$

The space on the right hand side has the *product topology* defined by the Banach space structure on $\mathcal{C}_U(G)$. The space $m_B(G)$ is a locally convex space of measures with the relative topology. A system of seminorms for the *product topology* on $m_B(G)$ is given by the family $\{\| \cdot \|_f\}_{f \in K(G)}$, where, for a function $f \in K(G)$,

$$\|\mu\|_f = \|f * \mu\|, \text{ for all } \mu \in m_B(G).$$

Definition 2.4. [8] A function $g \in \mathcal{C}(G)$ is called an *almost periodic function on G* if the family of translates of g, $\{g_a : a \in G\}$ is relatively compact in the sense of uniform convergence on G.

Let us denote by $AP(G)$ the set of all almost periodic functions on G.

Theorem 2.1. [8] Let G be a Hausdorff σ-compact, locally compact Abelian group. There exists an increasing sequence $(H_n)_{n \in N}$ of open, relatively compact subsets of G with $G = \cup_{n=1}^{\infty} H_n$ such that, for each $x \in G$,

$$\lim_{n \to \infty} \frac{\lambda(xH_n \Delta H_n)}{\lambda(H_n)} = 0.$$

In what follows, we consider a sequence $(H_n)_{n \in N}$ whose existence is assured by **Theorem 2.1.**

Let us consider

$$PAP_0(G) = \{\varphi \in \mathcal{C}(G) : \lim_{n \to \infty} \frac{1}{\lambda(H_n)} \int_{H_n} \mid \varphi(x) \mid d\lambda(x) = 0\}.$$

Definition 2.5. [3] A function $f \in \mathcal{C}(G)$ is called a *pseudo almost periodic function* if $f = g + \varphi$, where $g \quad AP(G)$ and $\varphi \quad PAP_0(G)$.

Denote by $PAP(G)$ the set of all pseudo almost periodic functions defined on the group G.

Theorem 2.2. [3] If $f \in PAP(G)$, then the limit

$$\lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} f(x) d\lambda(x)$$

exists and is finite.

Definition 2.6. [3] For $f \in PAP(G)$, we introduce the *mean value of the function* f, denoted by $M(f)$, as the limit

$$M(f) = \lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} f(x) d\lambda(x).$$

Denote by $\hat{G}$ the group of characters of the group G.

Proposition 2.1. [3] If $f \in PAP(G)$, there exists an at most countable set of characters of the group G, denoted by $\{\gamma_n \in \hat{G} : n \in \mathbb{N}\}$, such that $M(f\overline{\gamma}_n) \neq 0$, for $n \in \mathbb{N}$.

Definition 2.7. [3] Let f be a function in $PAP(G)$ and let $\{\gamma_n \in \hat{G} : n \in \mathbb{N}\}$ be the set of characters of the group G, such that $M(f\overline{\gamma}_n) \neq 0$, $n \in \mathbb{N}$. We define the *Fourier series of the pseudo almost periodic function* f by

$$\sum_{n=1}^{\infty} a_n \gamma_n(x), \quad x \in G$$

where

$$a_n = M(f \cdot \overline{\gamma}_n), \quad n \in \mathbb{N}.$$

The complex numbers a_n, $n \in \mathbb{N}$, are called the *Fourier coefficients of the function* f.

Theorem 2.3. [4] (Parseval's equality) Let f be a function in $PAP(G)$ and let

$$\sum_{n=1}^{\infty} a_n \gamma_n(x), \quad x \in G$$

be the Fourier series of the function f. Then we have

$$M(| f |^2) = \sum_{n=1}^{\infty} | a_n |^2 .$$

Theorem 2.4. [4] (Mean convergence) Let f be a function in $PAP(G)$ and let

$$\sum_{n=1}^{\infty} a_n \gamma_n(x), \quad x \in G$$

be the Fourier series of the function f. Then

$$\lim_{N\to\infty} M(\mid f - \sum_{k=1}^{N} a_k \gamma_k \mid^2) = 0.$$

Definition 2.8. [9] The measure $\mu \in m_B(G)$ is said to be an *almost periodic measure*, if for every $f \in K(G)$, we have $f * \mu \in AP(G)$.

The vector space of all almost periodic measures will be denoted by $ap(G)$.

Set

$$pap_0(G) = \{\mu \in m_B(G) :\ g * \mu \in PAP_0(G) \text{ for all } g \in K(G)\}.$$

Definition 2.9. [5] We say that $\mu \in m_B(G)$ is a *pseudo almost periodic measure*, if $\mu = \mu_{ap} + \mu_0$, where $\mu_{ap} \in ap(G)$ and $\mu_0 \in pap_0(G)$.

We denote by $pap(G)$ the set of all pseudo almost periodic measures on G.

Theorem 2.5. [5] Let μ be a pseudo almost periodic measure. Then there exists a unique complex number $M(\mu)$ such that $M(\mu) = M(f * \mu)$, for every function $f \in K(G)$ with $\int_G f(x) d\lambda(x) = 1$.

Definition 2.10. [5] Let μ be a pseudo almost periodic measure. We call the *mean value of the measure* μ and we denote it by $M(\mu)$, the complex number defined by

$$M(\mu) = M(f * \mu),$$

where $f \in K(G)$ with $\int_G f(x) d\lambda(x) = 1$.

Notation 2.1. For $\gamma \in \hat{G}$ and $g \in K(G)$ we denote

$$\hat{g}(\gamma) = \int_G \overline{\gamma}(x) g(x) d\lambda(x).$$

Definition 2.11. [2] By a *Fourier-Bohr series* we shall mean a formal sum

$$\sum_{\gamma \in \hat{G}} c_\gamma \gamma$$

such that the complex function $\gamma \in \hat{G} \to c_\gamma \in C$ has the property that, for every $g \in K(G)$,

$$\sum_{\gamma \in \hat{G}} \mid \hat{g}(\gamma) c_\gamma \mid^2 < \infty, \tag{1}$$

called the *summability property*.

The left hand side in (1) becomes

$$\sum_{n=1}^{\infty} \mid \lambda(\overline{\gamma}_n g) \mid^2 \mid M(\overline{\gamma}_n \mu) \mid^2$$

and using (2) we find that

$$\sum_{n=1}^{\infty} \mid \lambda(\overline{\gamma}_n g) \mid^2 \mid M(\overline{\gamma}_n \mu) \mid^2 = M(\mid g * \mu \mid^2) < \infty. \square$$

Definition 3.1. Let μ be a measure in $pap(G)$.The complex numbers $c_\gamma(\mu) = = M(\overline{\gamma}\mu)$, $\gamma \in \hat{G}$ are called the *Fourier-Bohr coefficients of* μ and the series

$$\sum_{\gamma \in \hat{G}} c_\gamma(\mu)\gamma,$$

is said to be the *Fourier-Bohr series of* μ.

Proposition 3.1. Let μ be a measure in $pap(G)$, $\mu = \mu_{ap} + \mu_0$, $\mu_{ap} \in ap(G)$, $\mu_0 \in pap_0(G)$. Then

a) $c_\gamma(\mu_0) = 0, \quad \gamma \in \hat{G}$;

b) $c_\gamma(\mu_{ap}) = c_\gamma(\mu), \quad \gamma \in \hat{G}$;

c) The Fourier-Bohr series of μ coincides with the Fourier-Bohr series of μ_{ap}.

Proof. a) For all $\gamma \in \hat{G}$, $\overline{\gamma}\mu_0 \in pap_0(G)$ [6, Theorem 2.2]. Thus $M(\overline{\gamma}\mu_0) = 0$ and this means $c_\gamma(\mu_0) = 0$, $\gamma \in \hat{G}$.

b) For all $\gamma \in \hat{G}$ we obtain

$$c_\gamma(\mu) = M(\overline{\gamma}\mu) = M(\overline{\gamma}\mu_{ap}) + M(\overline{\gamma}\mu_0) = M(\overline{\gamma}\mu_{ap}) = c_\gamma(\mu_{ap}).$$

c) The two series of μ and μ_{ap} coincide because they have the same Fourier-Bohr coefficients.$\square$

Notation 3.1. Let f be a function in $K(G)$. We denote by f', the function $f' : G \to C$, $f'(x) = f(x^{-1})$. We have that $f' \in K(G)$ [7, p. 406].

The next corollary immediately follows:

Corrolary 3.1. Let μ be a measure in $pap_0(G)$. Then the Fourier-Bohr series of μ is the null series.

Theorem 3.1. Let μ be a measure in $pap(G)$. Then μ is in $pap_0(G)$ if and only if $c_\gamma(\mu) = 0$ for all $\gamma \in \hat{G}$.

Proof. The necessity follows from **Proposition 3.1** a).

We now prove the sufficiency. Suppose that μ is in $ap(G)$, $\mu \neq 0$, and consider a function f in $K(G)$. Then $f * \mu \in AP(G)$.

For all $\gamma \in \hat{G}$ we obtain that

$$M[\overline{\gamma}(f * \mu)] = \lambda(\overline{\gamma} f) c_\gamma(\mu) = 0.$$

It follows that the Fourier coefficients of the function $f * \mu$ are null, and moreover the Fourier series of the function $f * \mu$, is the null series.

We obtain that $f * \mu = 0$ [2, p.138]. Thus $f * \mu = 0$ for all $f \in K(G)$. Moreover

$$\mu(f) = \int_G f(x) d\mu(x) = \int_G f'(ex^{-1}) d\mu(x) = f' * \mu(e) = 0.$$

Finally, we remark that $\mu(f) = 0$ for all $f \in K(G)$. This means that $\mu = 0$ and this contradicts our hypothesis that $\mu \neq 0$. It follows that $\mu \in pap_0(G)$. □

Notation 3.2. For $\gamma \in \hat{G}$ and $\nu \in m_F(G)$ we denote

$$\hat{\nu}(\gamma) = \int_G \overline{\gamma}(x) d\nu(x).$$

Theorem 3.2. Let μ be a measure in $pap(G)$ and ν a measure in $m_F(G)$. Then the pseudo almost periodic measure $\nu * \mu$ has the following Fourier-Bohr series:

$$\sum_{\gamma \in \hat{G}} \hat{\nu}(\gamma) c_\gamma(\mu) \gamma.$$

Proof. Let γ be in $\hat{G}$. Then $\overline{\gamma}\nu \in m_F(G)$. We also have that $\overline{\gamma}\mu \in pap(G)$ [6, Corollary 2.1] and $\overline{\gamma}\nu * \overline{\gamma}\mu \in pap(G)$ [6, Theorem 2.1]. Let f be a function in $K(G)$ such that $\lambda(f) = 1$.

We obtain:

$$M[\overline{\gamma}(\nu * \mu)] = M(\overline{\gamma}\nu * \overline{\gamma}\mu) = M[f * (\overline{\gamma}\nu * \overline{\gamma}\mu)] = M[\overline{\gamma}\nu * (f * \overline{\gamma}\mu)].$$

The third equality follows from Theorem 1.2 [1].

$$M[\overline{\gamma}\nu * (f * \overline{\gamma}\mu)] = \lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} [\overline{\gamma}\nu * (f * \overline{\gamma}\mu)](x) d\lambda(x) =$$

$$= \lim_{n\to\infty} \frac{1}{\lambda(H_n)} \int_{H_n} \int_G (f * \overline{\gamma}\mu)(xy^{-1}) \overline{\gamma}(y) d\nu(y) d\lambda(x).$$

Using Fubini's Theorem we obtain, for all $n \in I\!N$,

$$\frac{1}{\lambda(H_n)} \int_{H_n} \int_G (f * \overline{\gamma}\mu)(xy^{-1}) \overline{\gamma}(y) d\nu(y) d\lambda(x) =$$

$$= \int_G \overline{\gamma}(y) \left[\frac{1}{\lambda(H_n)} \int_{H_n} (f * \overline{\gamma}\mu)(xy^{-1}) d\lambda(x) \right] d\nu(y). \tag{3}$$

We consider the sequence of continuous functions $(F_n)_{n\in\mathbb{N}}$, where for all $n \in \mathbb{N}$, $F_n : G \to C$, and

$$F_n(y) = \frac{1}{\lambda(H_n)}\int_{H_n}(f * \overline{\gamma}\mu)(xy^{-1})d\lambda(x), \quad y \in G.$$

On the other hand, for all $n \in \mathbb{N}$ we have

$$\|F_n\| = \sup_{y\in G} \mid \frac{1}{\lambda(H_n)}\int_{H_n}(f * \overline{\gamma}\mu)(xy^{-1})d\lambda(x) \mid \leq \|f * \overline{\gamma}\mu\| < \infty,$$

and the constant function $y \to \|f * \overline{\gamma}\mu\|$ $(y \in G)$, is integrable with respect to the measure $\overline{\gamma}\nu \in m_F(G)$.

Since

$$\lim_{n\to\infty} F_n(y) = \lim_{n\to\infty}\frac{1}{\lambda(H_n)}\int_{H_n}(f * \overline{\gamma}\mu)(xy^{-1})d\lambda(x) = M(f * \overline{\gamma}\mu) = M(\overline{\gamma}\mu) = c_\gamma(\mu),$$

from (3) and from Lebesgue's Dominated Convergence Theorem we find that

$$M[\overline{\gamma}(\nu * \mu)] = \lim_{n\to\infty}\int_G \overline{\gamma}(y)\left[\frac{1}{\lambda(H_n)}\int_{H_n}(f * \overline{\gamma}\mu)(xy^{-1})d\lambda(x)\right]d\nu(y) =$$

$$= \int_G \overline{\gamma}(y)\left[\lim_{n\to\infty}\frac{1}{\lambda(H_n)}\int_{H_n}(f * \overline{\gamma}\mu)(xy^{-1})d\lambda(x)\right]d\nu(y) =$$

$$= c_\gamma(\mu)\int_G \overline{\gamma}(y)d\nu(y) = c_\gamma(\mu)\hat{\nu}(\gamma).\Box$$

Theorem 3.3. Let μ be a measure in $pap(G)$ and f a function in $K(G)$. Consider $\{\gamma_n \in \hat{G} : n \in \mathbb{N}\}$ the set of characters such that $M[\overline{\gamma}_n(f * \mu)] \neq 0$, $n \in \mathbb{N}$. Then

$$\lim_{N\to\infty} M[\mid f * \mu - \sum_{n=1}^{N}\hat{f}(\gamma_n)c_{\gamma_n}(\mu)\gamma_n \mid^2] = 0.$$

Proof. From **Theorem 2.4** we obtain

$$\lim_{N\to\infty} M[\mid f * \mu - \sum_{k=1}^{N} M[\overline{\gamma}_n(f * \mu)]\gamma_n \mid^2] = 0. \tag{4}$$

Combining (4) with the equality (2) obtained in the proof of **Lemma 3.1** namely,

$$M[\overline{\gamma}_n(f * \mu)] = \hat{f}(\gamma_n)c_{\gamma_n}(\mu),$$

our statement follows.□

Theorem 3.4. Let ν be a measure in $m_F(G)$ and μ a pseudo almost periodic measure having the Fourier-Bohr series

$$\sum_{\gamma\in\hat{G}} c_\gamma(\mu)\gamma.$$

Let f be a pseudo almost periodic function with the Fourier series

$$\sum_{n=1}^{\infty} a_n \gamma_n.$$

Then the following properties hold:

a) The series

$$\sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\gamma_n$$

converges to a measure $\tau \in ap(G)$ in the product topology;

b) The series

$$\sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\hat{\nu}(\gamma_n)\gamma_n$$

converges to a measure $\theta \in ap(G)$ in the product topology;

c) $\nu * \tau = \theta$.

Proof. a) We have the property

$$\sum_{n=1}^{\infty} | a_n |^2 = M(| f |^2) < \infty,$$

hence we can use Lemma 8.1 [2, p.133] and the statement follows;

b) For every $n \in I\!N$ we obtain

$$| \hat{\nu}(\gamma_n) | = | \int_G \overline{\gamma}_n(x) d\nu(x) | \leq | \nu | (G) < \infty.$$

Therefore

$$\sum_{n=1}^{\infty} | a_n \hat{\nu}(\gamma_n) |^2 \leq [| \nu | (G)]^2 \sum_{n=1}^{\infty} | a_n |^2 = [| \nu | (G)]^2 M(| f |^2).$$

Hence we can also use Lemma 8.1 [2, p.133] and the statement follows;

c) We shall prove that $c_\gamma(\nu * \tau) = c_\gamma(\theta)$ for every $\gamma \in \hat{G}$. Let γ be in $\hat{G}$. It follows from **Theorem 3.2** that $c_\gamma(\nu * \tau) = \hat{\nu}(\gamma) c_\gamma(\tau)$. We have $\overline{\gamma}\tau \in ap(G)$ [9, p. 84].

We can see that the series

$$\sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\overline{\gamma}\gamma_n$$

is convergent in the product topology, to the measure $\overline{\gamma}\tau \in ap(G)$.

One deduces that, for $g \in K(G)$,

$$\overline{\gamma}\tau(g) = \tau(g\overline{\gamma}) = (g\overline{\gamma})' * \tau(e) = g'\gamma * \tau(e).$$

Using the convergence of the series $\sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\gamma_n$ to the measure $\tau \in ap(G)$ in the product topology, and the fact that $g'\gamma \in K(G)$, we find:

$$g'\gamma * \tau(e) = \sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)[g'\gamma * \gamma_n(e)] =$$

$$= \sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\widehat{g'\gamma}(\gamma_n), \quad g \in K(G).$$

Therefore,

$$\overline{\gamma}\tau(g) = \sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\widehat{g^t\gamma}(\gamma_n), \quad g \in K(G).$$

Furthermore, for $g \in K(G)$, it follows

$$g * \overline{\gamma}\tau = \sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\widehat{g\gamma}(\gamma_n)\overline{\gamma}\gamma_n.$$

On the other hand we have

$$\left| \sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\widehat{g\gamma}(\gamma_n)\overline{\gamma}\gamma_n \right| \leq \sum_{n=1}^{\infty} | a_n | \, | c_{\gamma_n}(\mu) | \, | \widehat{g\gamma}(\gamma_n) | \leq$$
$$\leq \left[\sum_{n=1}^{\infty} | a_n |^2\right]^{1/2} \left[\sum_{n=1}^{\infty} | c_{\gamma_n}(\mu) |^2 \, | \widehat{g\gamma}(\gamma_n) |^2\right]^{1/2}.$$

Taking into account that $g\gamma \in K(G)$ and that the Fourier-Bohr series of μ satisfies (1) we obtain that

$$\sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu) \, \widehat{g\gamma}(\gamma_n)$$

is absolutely convergent and that the convergence of the series

$$\sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu) \, \widehat{g\gamma}(\gamma_n)\overline{\gamma}\gamma_n$$

is uniform. This shows the convergence of the series

$$\sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\overline{\gamma}\gamma_n$$

in the product topology, to $\overline{\gamma}\tau \in ap(G)$.

Thus,

$$\overline{\gamma}\tau = \sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu)\overline{\gamma}\gamma_n.$$

Using the continuity of M on $ap(G)$ [9, Corollaire 1, p. 82] we obtain

$$c_\gamma(\tau) = M(\overline{\gamma}\tau) = M(\lim_{n\to\infty}\sum_{k=1}^{n} a_k c_{\gamma_k}(\mu)\overline{\gamma}\gamma_k) = \sum_{n=1}^{\infty} a_n c_{\gamma_n}(\mu) M(\overline{\gamma}\gamma_n).$$

For $\alpha \in \hat{G}$ and for $x \in G$ we see that

$$M(\alpha) = M(\alpha_x) = M[\alpha(x)\alpha] = \alpha(x)M(\alpha).$$

Hence,

$$M(\alpha) = \begin{cases} 0, & \text{if } \alpha \in \hat{G}, \ \alpha \neq 1; \\ 1, & \text{if } \alpha = 1. \end{cases}$$

It is obvious that $\overline{\gamma}\gamma_n \in \hat{G}$, $n \in \mathbb{N}$. Therefore, for $n \in \mathbb{N}$, we have

$$M(\overline{\gamma}\gamma_n) = \begin{cases} 1, & \text{if } \gamma \in \{\gamma_k \in \hat{G} : \ k \in \mathbb{N}\}, \ \gamma = \gamma_n; \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

This means that, if $\gamma \in \{\gamma_k \in \hat{G} : k \in \mathbb{N}\}$, $c_\gamma(\nu * \tau) = \hat{\nu}(\gamma_p)\, a_p c_{\gamma_p}(\mu)$, where $p \in \mathbb{N}$ is such that $\gamma = \gamma_p$, while in the other case, $c_\gamma(\nu * \tau) = 0$.

We similarly deduce that $c_\gamma(\theta)$ has the same value to that of $c_\gamma(\nu * \mu)$.

Hence, one finds that the measures θ and $\nu * \tau$ are in $ap(G)$ and they have the same Fourier-Bohr coefficients. It follows from Corollary 8.2 [2, p. 140] that $\nu * \tau = \theta$. □

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