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STABILIZATION OF BILINEAR SYSTEMS BY MEANS OF A PRIORI BOUNDED CONTROLS

Dan LAMBADARIE

I.Introduction.

To get around the problem of stabilization for nonlinear systems two main strategies have been proposed:

(i) Asymptotic stabilization by means of a discontinous feedback law -see the pioneer work by H. Sussmann ([1]);

(ii) Asimptotic stabilization by means of a continous periodic time-varying feedback law - see the pioneer work by Sontag and Sussmann ([2]).

We can also notice the results obtained by J.M. Coron and J.B. Pomet (see [3],[4]) in stabilization of affine control systems $(\frac{dx}{dt} = \sum_{i=1}^{m} u_i g_i(x), x(0) = x_0 \in R^n)$, under a sufficiently strong hypothesis imposed to the Lie algebra determinated by the vector fields $g_1, \ldots, g_m$: $\dim L(g_1, \ldots, g_m) = n$, for $x \in R^n \setminus \{0\}$.

In the previous paper (see [5]) we show how much the controllability condition can be weakned , while still providing the possibility of stabilization . It is proved a result concerning stabilization with respect to a target set for affine control systems in the lack of the full rank condition .

Let consider the bilinear system (a particular case of affine system) :

(1) $$\frac{dx}{dt}=\sum_{i=1}^{m} u_i A_i x,\ x(0)=x_0\in S_\rho(0)\subset R^n,\ t\geq 0,\ A_i\in \mathcal{L}(R^n,R^n)\quad .$$

The goal of this paper is to show that the system (1) is asymptotically stabilizable with respect to a target set $V\subset S_\rho(0)$ by some admissible controls $u_i(t;x_0)$ that fulfill $|u_i(t;x_0)|\leq C$ where C is an a priori fixed positive constant.

II.Asimptotically stabilization with respect to a target set.

We take $u_i(t;x_0)$, i=1,...,m as an admissible controls if they lead to corresponding solution $x(t;x_0)$ in (1) satisfying $x(t;x_0)\in S_\rho(0)$, for any $t\geq 0$ and $x_0\in S_\rho(0)$. Consequently we may consider the ω-limit set :

(2) $$\Omega_+(x_0)\overset{def}{=}\left\{x\in R^n / x=\lim_{n\to\infty} x(t_n;x_0),\ \text{for some}\{t_n\}\uparrow\infty\right\}.$$

If $V\subset S_\rho(0)$ is a given set , one says that the system (1) is asymptotically stabilizable with respect to the target set V if there exist admissible controls $u_i(t;x_0)$, i=1,...,m. such that $\Omega_+(x_0)\subset V$ and $u_i(t;x_0)=0$ for any $x_0\in V$,i=1,...,m and $t\geq 0$.

III.The main result.

For symplyfying the writting and for a better understanding of the control construction the main result will be stated in the particular case m=2 and C=1 . Then let consider the bilinear system :

(3) $$\frac{dx}{dt}=u_1 A_1 x+u_2 A_2 x,\ x(0)=x_0\in S_\rho(0)\subset R^n,\ t\geq 0,\ A_1,A_2\in \mathcal{L}(R^n,R^n)$$

and denote : $ad\,A_1(A_2)\overset{def}{=}-A_1A_2+A_2A_1$, $ad^k A_1(A_2)=adA_1(ad^{k-1}A_1(A_2)),\ k\geq 2$.

The Lie algebra L(A_1,A_2) is a finite dimensional linear space and

$$S_1^2\overset{def}{=}\text{span}\left\{ad^k A_1(A_2)/k\geq 0\right\},\ S_2^1\overset{def}{=}\text{span}\left\{ad^k A_2(A_1)/k\geq 0\right\}$$

are finitely generated over R subspaces , that is :

there exist $\{Y_1,\dots,Y_{N_1}\}\subset S_1^2$,such that

$(\forall) A\in S_1^2,\ A=\sum_{k=1}^{N_1}\alpha_k Y_k,\ \alpha_k\in R,\ Y_k=ad^{k-1}A_1(A_2),\ k=1,\dots,N_1$ and

there exist $\{Z_1,\dots,Z_{N_2}\}\subset S_2^1$ such that

$(\forall) A\in S_2^1,\ A=\sum_{k=1}^{N_2}\beta_k Z_k,\ \beta_k\in R,\ Z_k=ad^{k-1}A_2(A_1),\ k=1,\dots,N_2$.

Theorem. *Let C>0 be a real constant. Then there exist the control $u=\begin{pmatrix}u_1\\u_2\end{pmatrix}, |u_i(t;x_0)|\le C,\ i=1,2$ such that the system* (3) *is asymptotically stabilizable with respect to the target set :* $V\overset{def}{=}\{x\in S_0\ (0)/ <x,Y_i x>=0,\ i=1,\dots,N_1,\ <x,Z_j x>=0,\ j=1,\dots N_2\}$.

Sketch of proof

First step: (Controls construction) .

Let C=1 and let T>0 be an arbitrary constant. We shall construct the controls on the interval $[lT,(l+1)T]$, $l\in N^*$. .

On the interval $[0,T]$:

$$(4)\ \begin{cases} u(t;x_0)=\tilde{u}(t;x_0)+\varepsilon\bar{u}(t;v_1(x_0),v_2(x_0)),\ \text{where}\ \tilde{u}=\begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\end{pmatrix},\ \bar{u}=\begin{pmatrix}\bar{u}_1\\ \bar{u}_2\end{pmatrix},\ \varepsilon\in(0,1),\\ \tilde{u}_1\overset{def}{=}\begin{cases}1, t\in[0,T/4]\\ -1, t\in[T/4,T/2]\\ 0, t\in[T/2,T]\end{cases},\quad \tilde{u}_2\overset{def}{=}\begin{cases}0, t\in[0,T/2]\\ 1, t\in[T/2,3T/4]\\ -1, t\in[3T/4,T]\end{cases}\\ \bar{u}_1(t;v_2(x_0))\overset{def}{=}\begin{cases}0 & ,t\in[0,3T/4]\\ <b_2(t),v_2(x_0)>, & t\in(3T/4,T]\end{cases},\quad \bar{u}_2(t;v_1(x_0))\overset{def}{=}\begin{cases}<b_1(t),v_1(x_0)>, & t\in[0,T/4]\\ 0 & ,t\in(T/4,T]\end{cases}\\ b_1(t)\overset{def}{=}(\exp tB_1)e_1,\ B_1=col[e_2,e_3,\dots,e_{N_1},\alpha^1],\ \text{with}\ (e_1,e_2,\dots,e_{N_1})\ \text{the canonical base of}\\ R^{N_1}\ \text{and}\ \alpha^1\in R^{N_1}\ \text{having the components}\ \alpha_i^1, i=1,\dots,N_1\ \text{defined by}\ ad^{N_1}A_1(A_2)=\sum_{i=1}^{N_1}\alpha_i^1 Y_i\\ b_2(t)\overset{def}{=}(\exp tB_2)\hat{e}_1,\ B_2=col[\hat{e}_2,\hat{e}_3,\dots,\hat{e}_{N_2},\alpha^2]\ \text{with}\ (\hat{e}_1,\hat{e}_2,\dots,\hat{e}_{N_2})\ \text{the canonical base of}\\ R^{N_2}\ \text{and}\ \alpha^2\in R^{N_2}\ \text{having the}\\ \text{components}\ \alpha_j^2, j=1,\dots,N_2\ \text{defined by}\ ,ad^{N_2}A_2(A_1)=\sum_{i=1}^{N_2}\alpha_j^2 Z_j\\ v_1(x_0)\overset{def}{=}-\begin{pmatrix}<x_0,\ Y_1x_0>\\ <x_0,\ Y_2x_0>\\ \dots\dots\\ <x_0,Y_{N_1}x_0>\end{pmatrix},\quad v_2(x_0)\overset{def}{=}-\begin{pmatrix}<x_0,\ Z_1x_0>\\ <x_0,\ Z_2x_0>\\ \dots\dots\\ <x_0,Z_{N_2}x_0>\end{pmatrix}\end{cases}$$

Denote $x(t;x_0)$, $t \in [0,T]$ the solution in (3) corresponding to u defined in (4).

$|u_i(t;x_0)| \le 1$ and $x(T;x_0) \in S_\rho(0)$ for $\varepsilon \in (0,1)$ sufficiently small.

Henceforth, $x(T;x_0)$ will take the place of x_0. The control $\hat{u}(t;x_0)$ on the next interval $[T,2T]$ is given by :

(5) $\hat{u}(t;x_0) \stackrel{\text{def}}{=} u(t-T;x(T;x_0)), t \in [T,2T]$

and it generates $x(t;x_0)$, the solution in (3) for $t \in [T,2T]$, $x(2T;x_0) \in S_\rho(0)$ and the process is iterated.

If $x(t;x_0), t \ge 0$ is the solution in (3) periodically defined on each interval $[lT,(l+1)T]$, $l \in N$ then generally the control is :

(6) $\hat{u}(t;x_0) \stackrel{\text{def}}{=} u(t-lT;x(lT;x_0)), t \in [lT,(l+1)T], l \in N$.

<u>Second step</u> ; (The estimate for the solution in (3) at the moment t=T).

Let define $M_1 \stackrel{\text{def}}{=} \int_0^{T/4} b_1(t)\, b_1^*(t)\, dt$, $M_2 \stackrel{\text{def}}{=} \int_{3T/4}^{T} b_2(t)\, b_2^*(t)\, dt$.

M_1 and M_2 are $(N_1 \times N_1)$ respectively $(N_2 \times N_2)$ positive-definite matrix.

The solution $x(t;x_0)$ of (3) considered on $[0,T]$ satisfies in t = T the representation :

(7) $x(T;x_0) = x_0 + \varepsilon \operatorname{col}[Y_1 x_0, \ldots, Y_{N_1} x_0, Z_1 x_0, \ldots, Z_{N_2} x_0]\, M\, v(x_0) + o(\varepsilon)\|v(x_0)\|^2$ where

$\varepsilon \in (0,1), M = \operatorname{diag}(M_1, M_2), v(x_0) \in R^{N_1+N_2}, v(x_0) = \begin{pmatrix} v_1(x_0) \\ v_2(x_0) \end{pmatrix}$ (see (4)) and $\lim_{\varepsilon \to 0} \frac{o(\varepsilon)}{\varepsilon} = 0$.

Using (7) and the above considerations we obtain the estimate :

$\|x(T;x_0)\|^2 \le \|x_0\|^2 - \gamma_0 \|v(x_0)\|^2$, where $\gamma_0 = \varepsilon \min(\gamma_1, \gamma_2)$ with $\gamma_i > 0$ is the smallest eigenvalue of M_i i=1,2 .

<u>Third step</u> ;

By an iterative process one can obtain the estimate:

$\|x(lT;x_0)\|^2 \le \|x((l-1)T;x_0)\|^2 - \gamma_0 \|v(x((l-1)T;x_0))\|^2$.

Taking the sum of these inequalities , it follows :

$$\|x(lT;x_0)\|^2 \le \|x_0\|^2 - \gamma_0 \sum_{k=0}^{l-1} \|v(x(kT;x_0))\|^2 \ .$$

Consequently ,

$$\lim_{l\to\infty} \|v(x(lT;x_0))\|^2 = 0 \quad \Leftrightarrow \quad \begin{cases} \lim\limits_{l\to\infty} < x(lT;x_0), Y_i x(lT;x_0) >= 0 , i = 1,\ldots, N_1 \\ \lim\limits_{l\to\infty} < x(lT;x_0), Z_j x(lT;x_0) >= 0 , j = 1,\ldots, N_2 \end{cases} .$$

If x^* is a limit point of $\{x(lT;x_0)\}_{l\ge 0} \subset S_\rho(0)$, then :

$< x^*, Y_i x^* >= 0 , i = 1,\ldots, N_1$, $< x^*, Z_j x^* >= 0 , j = 1,\ldots, N_2$ and it follows $x^* \in V$.

Finally , we adjust the control (defined in (4)-(6)) multiplying it on the interval $[lT,(l+1)T]$ $l \in N$ by $\|v(x(lT;x_0))\|^2$ and so any point of V is a stationary one (see $\|v(x)\|^2 = 0 \Leftrightarrow x \in V$) .

This adjustment doesn't change essentially the above results and it send us to the same conclusion $x^* \in V$.

Any string $\{t_n\}_{n\in N} \uparrow \infty$ can be written as $t_n = nT + t_n^*, n \in N, t_n^* \in [0,T)$ and by continuous dependence of a solution on parameters we get $\lim\limits_{n\to\infty} x(t_n;x_0) \in V$, then the proof is complete.

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Department of Mathematics
Military Technical Academy
Blvd. Regina Maria nr. 81-83
Bucharest , Romania.